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A universal identifier for communication channels

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Abstract
Pseudo-differential operators, viewed as superpositions of time-frequency shifts, are natural models for communication channels. Channel identification is thus to find a proper input signal that induces an injective map on certain spaces of pseudo-differential operators. It is known that this is possible for channels with finite energy and spreading support area less than 1 (resp. 1/2) if the location and shape of the support area is known (resp. unknown) and the identifier depends on the spreading support. We will construct a universal input signal, which is independent of the spreading support, that identifies all such spaces of channels. The novelty of this result lies in the universality of this identifier.

Keywords Pseudo-differential operators · Channel identification · Time-frequency analysis · Gabor analysis

Mathematics Subject Classification 47G30 · 42B35 · 42C35

1 Introduction

A communication channel essentially acts on an input signal by time shifts (due to distances) and frequency shifts (i.e., the Doppler effect due to motions), both superposed with proper weights (due to various factors) which are representable by some function (or distribution) η. For this reason it is natural to model a communication channel by a pseudo-differential operator H_η of the following form (See e.g., [1–4]):

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$$(H_\eta f)(x) = \iint_{\mathbb{R}^2} \eta(t, v)(M_v T_t f)(x)\, dv\, dt, \tag{1}$$

where $T_t : f(x) \mapsto f(x - t)$ is the time shift (translation) operator and $M_v : f(x) \mapsto e^{2\pi i v \cdot x} f(x)$ is the the frequency shift (modulation). See [5,6] for the derivation of this particular form, and [7, Chapter 14.1], [8, Chapter 2.2] for other forms.

The weight η is called the spreading function (although it may also be a distribution) of the operator and its support is accordingly called the spreading support. The domain of H_η and the scope of η considered in this article will be specified later. In communication theory, if the spreading support is inside a rectangle of unit area, then the channel is called underspread, otherwise it is called overspread. Here, as is also the case in some literature (e.g., [9] etc.), we borrow these concepts, but define them a bit differently: we say a channel (or its spreading support) is overspread if the spreading support is compact Jordan measurable with area larger than 1; and similarly we say it is underspread if the spreading support is compact Jordan measurable with area less than 1.

Let $U \subset \mathbb{R}^2$ be a compact Jordan measurable set. Through (1) each member in $L^2(U)$ models a communication channel whose spreading support is inside U with finite energy. For a fixed input signal g, we define a linear map Φ_g to be

$$\Phi_g : \eta \mapsto H_\eta g.$$

If Φ_g is bounded from $L^2(U)$ to $L^2(\mathbb{R})$, then we call it the identification map, call g the identifier, and call $\Phi_g \eta$ the response. If Φ_g is injective on $L^2(U)$, then we say $L^2(U)$ (as well as the space of pseudo-differential operators or communication channels with $L^2(U)$ spreading functions) is identifiable.

In the late 1950s, Kailath [10] proclaimed that a collection of unknown communication channels having common maximum delay a and common maximum Doppler shift b would be identifiable by a single input signal if and only if $ab \leq 1$, i.e., if and only if the spreading function η is supported in a rectangle of area at most 1. It was later shown (see [5]) that a channel has to be underspread (in our tweaked definition) to be identifiable, and explicit reconstruction methods were given (see [9]), see also [11] for relevant work from an alternative perspective, and [12] for blind identification (meaning the support of η is unknown).

Our main result is the construction of a universal identifier that identifies all such underspread channels:

Theorem 1 *There exists a tempered distribution g, such that for any compact Jordan measurable set $U \subset \mathbb{R}^2$ with area less than*

(i) 1 *if the location of U is known, or*
(ii) $1/2$ *if the location of U is unknown,*

Φ_g is bounded and injective from $L^2(U)$ to $L^2(\mathbb{R})$.

The universality means that the single identifier g works for all identifiable spaces, while in [9,12] one has to design a rectification scheme as well as an identifier based on the shape of U.

Using advanced time-frequency tools, we also give in the appendix a conciser exposition of the reconstruction method used in [9].

2 Preliminaries

To understand our construction, it is necessary to be familiar with existing techniques for identifying underspread channels. Unfortunately this is heavy in notation and intensive in computation, which is an obstacle for this topic to be accessible to people outside the community. In this section we only introduce prerequisite concepts, relevant details are in the appendix part.

Let $A(\mathbb{R})$ be the space of functions that are Fourier transforms of $L^1(\mathbb{R})$ functions, and $A'(\mathbb{R})$ be the space of distributions that are Fourier transforms of $L^\infty(\mathbb{R})$ functions. These are Banach spaces with norms [13]

$$\|\hat{f}\|_{A(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})}; \quad \|\hat{f}\|_{A'(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})}.$$

Let $\psi_0 \in C^\infty(\mathbb{R})$ be compactly supported in $[-\epsilon_0 - 1/2,\ \epsilon_0 + 1/2]$ for some small fixed ϵ_0 with

$$\psi_0(x) : \begin{cases} = 1 & x \in \left(-\frac{1}{2} + \epsilon_0,\ \frac{1}{2} - \epsilon_0\right), \\ \in [0, 1] & x \in \left[-\frac{1}{2} - \epsilon_0,\ -\frac{1}{2} + \epsilon_0\right] \cup \left[\frac{1}{2} - \epsilon_0, \frac{1}{2} + \epsilon_0\right], \end{cases}$$

in such a way that its shifts form a BUPU (bounded uniform partition of unity):

$$\sum_{k \in \mathbb{Z}} T_k \psi_0 \equiv 1.$$

Wiener-Amalgam spaces capture functions or distribution with different local and global topological properties, they are defined as

$$W^{A,\ell^p}(\mathbb{R}) = \{f : \|f\|_{W^{A,\ell^p}(\mathbb{R})} = \|\{\|T_k\psi_0 \cdot f\|_{A(\mathbb{R})}\}_{k\in\mathbb{Z}}\|_{\ell^p} < \infty\},$$

and

$$W^{A',\ell^q}(\mathbb{R}) = \{g : \|f\|_{W^{A',\ell^q}(\mathbb{R})} = \|\{\|T_k\psi_0 \cdot g\|_{A'(\mathbb{R})}\}_{k\in\mathbb{Z}}\|_{\ell^q} < \infty\}.$$

Obviously these norms depend on the choice of the window ψ_0, but different choices of ψ_0 induce equivalent norms [13], therefore corresponding underlying spaces do not change. Of special interests to us are $W^{A,\ell^1}(\mathbb{R})$ and $W^{A',\ell^\infty}(\mathbb{R})$, which respectively coincides with the Feichtinger algebra and its dual with equivalent norms (see [14,15] and [16, Theorem 3.2.6]). This equivalence is a direct consequence of [17, Theorem 4.1.2] and [18], alternatively see [19,20] and references there for more details and history over this equivalence. There is also the continuous embedding (S denotes the

Schwartz class and S' denotes tempered distributions):

$$S(\mathbb{R}) \overset{\text{dense}}{\subset} W^{A,\ell^1}(\mathbb{R}) \overset{\text{dense}}{\subset} L^2(\mathbb{R}) \subset W^{A',\ell^\infty}(\mathbb{R}) \subset S'(\mathbb{R}). \tag{2}$$

The Fourier transform F, and time-frequency shifts T_t, M_v are automorphisms on each of them [21]. Such a setting provides us with a way to work with norms instead of semi-norms. See [22–26] for more information on the Feichtinger algebra and the Banach Gelfand triple $W^{A,\ell^1}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset W^{A',\ell^\infty}(\mathbb{R})$, and see [27] for motivations. See also [28] and [29, Prop 2.4] for their connections to modulation spaces.

Denote δ_t as the Dirac measure at t. For $r > 0$ and $n \in \mathbb{N}$, we use the following two notations to denote an unweighted Dirac comb (also called delta train, spiked train, impulse train or Shah distribution in different literature) and a periodically weighted Dirac comb respectively:

$$g_r = \sum_{j\in\mathbb{Z}} \delta_{jr}, \qquad g_{\vec{c}} = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{n-1} c_k \delta_{\frac{nj+k}{\sqrt{n}}}.$$

where $\vec{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{C}^n$, i.e., deltas in $g_{\vec{c}}$ are supported on $\frac{1}{\sqrt{n}}\mathbb{Z}$ with corresponding weights being elements in $\vec{c}$ periodized, thus $\sqrt{n}$ is a period of $g_{\vec{c}}$. One should also notice that such a periodically weighted Dirac comb is just linear combinations of several (shifted) unweighted Dirac combs.

It is easy to see (by definition of Wiener-Amalgam norms) that they are both in $W^{A',\ell^\infty}(\mathbb{R})$ (thus they are also tempered distributions) with

$$\|g_r\|_{W^{A',\ell^\infty}(\mathbb{R})} \le 3\max\left(\frac{1}{r}, 1\right), \quad \|g_{\vec{c}}\|_{W^{A',\ell^\infty}(\mathbb{R})} \le 3\|\vec{c}\|_{\ell^1}. \tag{3}$$

With tools from Gabor analysis, one can decompose a communication channel into small sub-channels supported in rectangular boxes on the time-frequency plane. This can also be arranged for finite dimensional vector spaces, see e.g., [30–32] for the history and development of concepts introduced below.

On $\mathbb{C}^n$, define the discrete translation T and discrete modulation M to be

$$T = \begin{pmatrix} 0 & & & & 1 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \ddots & & \\ & & & \omega^{n-2} & \\ & & & & \omega^{n-1} \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{n}}$ is a primitive n-th root of unity. We are overloading the same notation for their continuous counterparts, but depending on whether they are applied to a function or a vector, the meaning should be clear.

Discrete shifts and discrete modulations commute up to a phase factor (just like their continuous counterparts) with $MT = \omega TM$ and $\{\omega, M, T\}$ together under usual multiplication generates a representation of the finite Heisenberg group (see e.g, [33]).

A discrete Gabor system on $\mathbb{C}^n$ takes the form $\{M^j T^k \vec{c}\}_{(j,k)\in\Gamma}$, where $\vec{c} \in \mathbb{C}^n$ is the window vector, and $\Gamma \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ is the support of this system. It is called a Gabor matrix and denoted as $G_\Gamma(\vec{c})$ if it is written into the matrix form with $M^j T^k \vec{c}$ being column vectors, the ordering of columns will be specified at the place where it makes a difference.

Discrete Gabor systems possess many nice properties, what we need in this article is that the set

$$\{\vec{c} \in \mathbb{C}^n : \{M^j T^k \vec{c}\}_{(j,k)\in\mathbb{Z}_n\times\mathbb{Z}_n} \text{has full spark}\},$$

is open dense in $\mathbb{C}^n$ [34,35]. Here having full spark means any n distinct vectors from $\{M^j T^k \vec{c}\}_{(j,k)\in\mathbb{Z}_n\times\mathbb{Z}_n}$ are linearly independent.

3 Main result

Lemma 1 *Let $n \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}_n \times \mathbb{Z}_n$, let $E_\Lambda^{(n)}$ be the union of cells in the grid as defined in (10) in the appendix, then there exists a dense subset $Y \subset \mathbb{C}^n$ such that if $\vec{c} \in Y$, then $\Phi_{g_{\vec{c}}}$ is injective on $L^2(E_\Lambda^{(n)})$ for all Λ with*

(i) $|\Lambda| \leq n$ *if the location of $E_\Lambda^{(n)}$ is known;*

(ii) $|\Lambda| \leq \lfloor \frac{n}{2} \rfloor$ *if the location of $E_\Lambda^{(n)}$ is unknown.*

Proof For a fixed known $\Lambda \subset \mathbb{Z}_n \times \mathbb{Z}_n$ with $|\Lambda| \leq n$, define the set Y_Λ to be

$$Y_\Lambda = \{\vec{c} \in \mathbb{C}^n : \Phi_{g_{\vec{c}}} \text{is injective on} L^2(E_\Lambda^{(n)})\}.$$

By Proposition 1 in the appendix and the main result in [35], Y_Λ is open dense in $\mathbb{C}^n$. Now since $\mathbb{Z}_n \times \mathbb{Z}_n$ is a finite set, there are finitely many Λ with $|\Lambda| \leq n$, therefore the intersection set

$$Y = \bigcap_{\substack{\Lambda\subset\mathbb{Z}_n\times\mathbb{Z}_n \\ |\Lambda|\leq n}} Y_\Lambda,$$

is not only non-empty but also dense in $\mathbb{C}^n$, and it is the desired set for (i).

Now if the location (the location also determines the shape) of $E_{\Lambda_1}^{(n)}, E_{\Lambda_2}^{(n)}$ is unknown and $|\Lambda_1|, |\Lambda_2| \leq \lfloor n/2 \rfloor$ (Λ_1, Λ_2 need not be distinct), then $|\Lambda_1 \cup \Lambda_2| \leq n$, therefore $\Phi_{g_{\vec{c}}}$ would be injective on $L^2(E_{\Lambda_1}^{(n)} \cup E_{\Lambda_2}^{(n)})$ for any $\vec{c} \in Y$, which means if $\eta_i \in L^2(E_{\Lambda_i}^{(n)})$ for $i = 1, 2$ (i.e., $\eta_1 - \eta_2 \in L^2(E_{\Lambda_1}^{(n)} \cup E_{\Lambda_2}^{(n)})$), then

$$\Phi_{g_{\vec{c}}} \eta_1 \neq \Phi_{g_{\vec{c}}} \eta_2,$$

and this establishes (ii). □

Theorem 1 *There exists a tempered distribution g, such that for any compact Jordan measurable set $U \subset \mathbb{R}^2$ with area less than*

(i) 1 *if the location of* U *is known, or*
(ii) $1/2$ *if the location of* U *is unknown,*

Φ_g *is bounded and injective from* $L^2(U)$ *to* $L^2(\mathbb{R})$.

Proof It suffices to show (i), then (ii) follows using the same argument as in the proof of Lemma 1. We will inductively construct the identifier first, and then show that it has the desired property. The fundamental idea is to start with a primitive candidate, then keep adding controlled perturbation to it when passing to the limit, so that on each rectification the constructed identifier can always be viewed as the sum of a body part (which is a usual identifier as in the appendix) and a tail part (which is a small perturbation dominated by the body part).
Construction:

(1) First, let us set up the initial case. Take an identifier $g_{\vec{c}^{(1)}}$ with $\vec{c}^{(1)} \in \mathbb{C}^4$ so that on the grid $E^{(4)}_{\mathbb{Z}_4\times\mathbb{Z}_4}$, $g_{\vec{c}^{(1)}}$ identifies $L^2(E^{(4)}_\Lambda)$ as long as $\Lambda \subset \mathbb{Z}_4 \times \mathbb{Z}_4$ with $|\Lambda| = 4$. Such $\vec{c}^{(1)}$ exists by Lemma 1. Denote

$$\sigma^{(1)}_{\min} = \min_{\substack{\eta\in L^2(E^{(4)}_\Lambda)\setminus\{0\}\\ |\Lambda|=4}} \frac{\|\Phi_{g_{\vec{c}^{(1)}}}\eta\|_{L^2(\mathbb{R})}}{\|\eta\|_{L^2(E^{(4)}_\Lambda)}}, \qquad \sigma^{(1)}_{\max} = \max_{\substack{\eta\in L^2(E^{(4)}_\Lambda)\setminus\{0\}\\ |\Lambda|=4}} \frac{\|\Phi_{g_{\vec{c}^{(1)}}}\eta\|_{L^2(\mathbb{R})}}{\|\eta\|_{L^2(E^{(4)}_\Lambda)}},$$

Here min and max are indeed attainable, since by Proposition 1 (in the appendix) they reduce to singular values of Gabor matrices $G_\Lambda(\vec{c}^{(1)})$, and there are finitely many such matrices as there are only finitely many such Λ. For convenience we may also without loss of generality assume that $\|\vec{c}^{(1)}\|_{\ell^1} = 1$, this can be done by scaling $\vec{c}^{(1)}$.

(2) Next, suppose that we have already constructed the identifier $g_{\vec{c}^{(k)}}$ with $\vec{c}^{(k)} \in \mathbb{C}^{4^k}$ such that, if $\Lambda \subset \mathbb{Z}_{4^k} \times \mathbb{Z}_{4^k}$ with $|\Lambda| = 4^k$, then $g_{\vec{c}^{(k)}}$ identifies $L^2(E^{(4^k)}_\Lambda)$, and denote

$$\sigma^{(k)}_{\min} = \min_{\substack{\eta\in L^2(E^{(4^k)}_\Lambda)\setminus\{0\}\\ |\Lambda|=4^k}} \frac{\|\Phi_{g_{\vec{c}^{(k)}}}\eta\|_{L^2(\mathbb{R})}}{\|\eta\|_{L^2(E^{(4^k)}_\Lambda)}}, \qquad \sigma^{(k)}_{\max} = \max_{\substack{\eta\in L^2(E^{(4^k)}_\Lambda)\setminus\{0\}\\ |\Lambda|=4^k}} \frac{\|\Phi_{g_{\vec{c}^{(k)}}}\eta\|_{L^2(\mathbb{R})}}{\|\eta\|_{L^2(E^{(4^k)}_\Lambda)}}.$$

(3) Now let us look at the $k+1$ case, we embed $\vec{c}^{(k)}$ into $\mathbb{C}^{4^{k+1}}$ by inserting three 0s after each entry of $\vec{c}^{(k)}$, and denote the resulting vector as $\vec{d}$, i.e.,

$$\vec{d} = \left(\vec{c}^{(k)}_0, 0, 0, 0, \vec{c}^{(k)}_1, 0, 0, 0, \ldots, \vec{c}^{(k)}_{4^k-1}, 0, 0, 0\right).$$

Let $r_k = \sqrt{4^k} = 2^k$, and set

$$a_{k+1} = \frac{1}{\sqrt{r_k}} \min\left(\frac{1}{\sqrt{r_k}},\ \frac{1}{3}\sigma^{(k)}_{\min},\ \frac{1}{3^2}\sigma^{(k-1)}_{\min}, \ldots,\ \frac{1}{3^k}\sigma^{(1)}_{\min}\right), \tag{4}$$

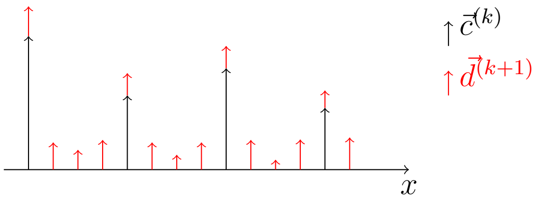

Fig. 1 Each $g_{\vec{c}^{(k+1)}}$ is a perturbation of its predecessor

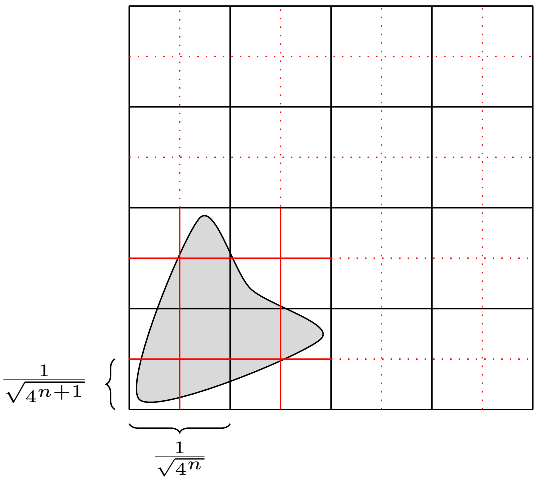

Fig. 2 Finer grid

with a_1 set to 1. Consider an ℓ^1 ball of radius a_{k+1} around $\vec{d}$, we pick a vector $\vec{c}^{(k+1)}$ from this ball so that if $\Lambda \subset \mathbb{Z}_{4^{k+1}} \times \mathbb{Z}_{4^{k+1}}$ with $|\Lambda| = 4^{k+1}$, then $g_{\vec{c}^{(k+1)}}$ identifies $L^2(E_\Lambda^{(4^{k+1})})$. Existence of such a vector is guaranteed by Lemma 1. Denote

$$\vec{d}^{(k+1)} = \vec{c}^{(k+1)} - \vec{d},$$

and set $\vec{d}^{(1)} = \vec{c}^{(1)}$. Repeat the above procedure, and look at

$$g = \sum_{k=1}^{\infty} g_{\vec{d}^{(k)}}. \tag{5}$$

We claim that g is the universal identifier (Fig. 1).

Verification:
Suppose $U \subset \mathbb{R}^2$ is a compact underspread Jordan measurable set. The compactness ensures that we can find an n large enough, such that there is an $E_{\Gamma_0}^{(4^n)}$ with $|\Gamma_0| = 4^n$ that covers U. Moreover, by simply splitting each cell in $E_\Gamma^{(4^n)}$ into four equal smaller cells, we get a new covering on the finer grid $E_{\mathbb{Z}_{4^{n+1}} \times \mathbb{Z}_{4^{n+1}}}^{(4^{n+1})}$. As shown below (Fig. 2):

Inductively carry out this splitting procedure, and denote the covering in the $\mathbb{Z}_{4^{n+j}} \times \mathbb{Z}_{4^{n+j}}$ $(j \geq 0)$ grid as $E_{\Gamma_j}^{(4^{n+j})}$. It is easy to see that they all satisfy the condition

$|\Gamma_j| = 4^{n+j}$ and correspond to the same set of area 1 and each induces a square Gabor matrix of size $4^{(n+j)} \times 4^{(n+j)}$, thus by the construction of g in (5) we get that

$$\sum_{k=1}^{\infty} \|\vec{d}^{(k)}\|_{\ell^1} \leq \sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} \frac{1}{r_{k-1}} = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2,$$

hence we apply (3) to get

$$\|g\|_{W^{A',\ell^\infty}(\mathbb{R})} = \left\| \sum_{k=1}^{\infty} g_{\vec{d}^{(k)}} \right\|_{W^{A',\ell^\infty}(\mathbb{R})} \leq \sum_{k=1}^{\infty} \|g_{\vec{d}^{(k)}}\|_{W^{A',\ell^\infty}(\mathbb{R})} \leq 3 \sum_{k=1}^{\infty} \|\vec{d}^{(k)}\|_{\ell^1} \leq 6,$$

which shows $g \in W^{A',\ell^\infty}(\mathbb{R})$, and is also a tempered distribution by (2).

Now on $L^2(E_{\Gamma_0}^{(4^n)})$, we shall view g as a body part which identifies the space plus a tail part which is a controlled perturbation that does not sabotage the identification:

$$\begin{aligned} g &= \underbrace{g_{\vec{d}^{(1)}} + g_{\vec{d}^{(2)}} + \cdots + g_{\vec{d}^{(n)}}}_{\text{body}} + \underbrace{g_{\vec{d}^{(n+1)}} + g_{\vec{d}^{(n+2)}} + \cdots}_{\text{tail}} \\ &= g_{\vec{c}^{(n)}} + \underbrace{g_{\vec{d}^{(n+1)}} + g_{\vec{d}^{(n+2)}} + \ldots,}_{\text{perturbation}} \end{aligned}$$

the body part (by the definition of $\vec{d}^{(k)}$) adds up to $g_{\vec{c}^{(n)}}$, which by our construction identifies $L^2(E_{\Gamma_0}^{(4^n)})$ with lower bound $\sigma_{\min}^{(n)}$ and upper bound $\sigma_{\max}^{(n)}$, while for each $g_{\vec{d}^{(n+j)}}$ $(j \geq 1)$ in the tail, recall that by (4) we have

$$\|\vec{d}^{(n+j)}\|_{\ell^1} \leq a_{n+j} \leq \frac{1}{3^k \sqrt{r^{n+j}}} \sigma_{\min}^{(n)}.$$

The Gabor matrix $G_{\Gamma_k}(\bar{d}^{(n+j)})$ is a square matrix and has 4^{n+j} columns, each column has same ℓ^1 norm as $\bar{d}^{(n+j)}$. Consequently we obtain

$$\|G_{\Gamma_j}(\bar{d}^{(n+j)})\| \leq \sqrt{4^{n+j}} \|\bar{d}^{(n+j)}\|_{\ell^1} \leq r_{n+j} a_{n+j} \leq \frac{\sqrt{r_{n+j}}}{3^j} \sigma_{\min}^{(n)}.$$

By Proposition 1 (in the appendix), we can conclude that if $\eta \in L^2(U) \subseteq L^2(E_{\Gamma_j}^{(4^{n+j})})$, then

$$\|\Phi_{g_{\vec{d}^{(n+j)}}} \eta\|_{L^2(\mathbb{R})} \leq \frac{1}{3^j} \sigma_{\min}^{(n)} \|\eta\|_{L^2(U)}.$$

Therefore the tail part is dominated by $\sigma_{\min}^{(n)}$:

$$\left\| \sum_{j=1}^{\infty} \Phi_{g_{\vec{d}(n+j)}} \eta \right\|_{L^2(\mathbb{R})} \leq \sum_{j=1}^{\infty} \| \Phi_{g_{\vec{d}(n+j)}} \eta \|_{L^2(\mathbb{R})} \leq \sum_{j=1}^{\infty} \frac{1}{3^j} \sigma_{\min}^{(n)} \|\eta\|_{L^2(U)}$$
$$= \frac{1}{2} \sigma_{\min}^{(n)} \|\eta\|_{L^2(U)}.$$

Consequently we get

$$\|\Phi_g \eta\|_{L^2(\mathbb{R})} = \left\| \left(\Phi_{g_{\vec{c}(n)}} + \sum_{j=1}^{\infty} \Phi_{g_{\vec{d}(n+j)}} \right) \eta \right\|_{L^2(\mathbb{R})} \geq \|\Phi_{g_{\vec{c}(n)}} \eta\|_{L^2(\mathbb{R})}$$
$$- \left\| \sum_{j=1}^{\infty} \Phi_{g_{\vec{d}(n+j)}} \eta \right\|_{L^2(\mathbb{R})} \geq \frac{1}{2} \sigma_{\min}^{(n)} \|\eta\|_{L^2(U)},$$

and

$$\|\Phi_g \eta\|_{L^2(\mathbb{R})} \leq \|\Phi_{g_{\vec{c}(n)}} \eta\|_{L^2(\mathbb{R})} + \left\| \sum_{j=1}^{\infty} \Phi_{g_{\vec{d}(n+j)}} \eta \right\|_{L^2(\mathbb{R})} \leq \left(\sigma_{\max}^{(n)} + \frac{1}{2} \sigma_{\min}^{(n)} \right) \|\eta\|_{L^2},$$

which shows Φ_g is upper and lower bounded on $L^2(U)$, and g is thus the universal identifier since U is arbitrary. □

Appendix: Rectification and discretization

This part contain exposition material of the method used in [9] to discretize the identification map on irregular compact underspread Jordan measurable sets. The decomposition formula in Proposition 1 below is only vaguely implied there but not explicitly stated. We adopted a different approach which uses the adjoint relation between the identification map and the short time Fourier transform, as a result, the derivation here should be simpler and clearer.

The Zak transform Z_r of a function $f \in L^2(\mathbb{R})$ is defined as

$$(Z_r f)(x, w) = \sum_{k \in \mathbb{Z}} f(x + kr) e^{-2\pi i kr \cdot w}, \tag{6}$$

where the right hand side is defined a.e. and $r > 0$ is a parameter.

The short time Fourier transform (STFT) V_ϕ with respect to a window ϕ on $\mathbb{R}^n$ can be written in several ways:

$$(V_\phi f)(t, v) = \langle f, M_v T_t \phi \rangle = e^{-2\pi i t \cdot v} \int_{\mathbb{R}} f(x + t) \overline{\phi(x)} e^{-2\pi i x \cdot v} \, dx. \tag{7}$$

The integral form is well defined for f, ϕ being $L^2(\mathbb{R})$ functions, while the bracket form can be applied to any dual pairing.

Comparing the definition in (6) and (7), one can see that g_r links STFT to Zak transforms (see also [6, Section 3.3]):

$$V_{g_r} f = e^{-2\pi i t\cdot v} Z_r f.$$

Moreover, for nice functions such as Schwartz class funcitons f, g, η, we have

$$\iiint_{\mathbb{R}^3} \eta(t,v)\overline{f(x)\overline{(M_v T_t g_r)(x)}}\, dx\, dv\, dt$$
$$= \iiint_{\mathbb{R}^3} \eta(t,v)(M_v T_t g_r)(x)\overline{f(x)}\, dv\, dt\, dx,$$

which can be further written as

$$\langle \eta, V_g f\rangle = \langle \Phi_g \eta, f\rangle, \tag{8}$$

under proper dual pairing. In particular [6, Theorem 4.1] (which is based on kernel theorems in [7, Chapter 14.4] and [36, Lemma 4.1]) shows that (8) holds for $g \in W^{A',\ell^\infty}(\mathbb{R})$, $f \in W^{A,\ell^1}(\mathbb{R})$ and $\eta \in L^2(U)$ where $U \subset \mathbb{R}^2$ is compact. See also [11,37].

Now for $\eta \in L^2(U)$, we can define $\Phi_{g_{\vec{c}}}$ to be the element in $L^2(\mathbb{R})$ that satisfies (8), then by the density in (2) we get that

$$\langle \Phi_{g_{\vec{c}}} \eta, f\rangle = \langle \eta, V_{g_{\vec{c}}} f\rangle, \tag{9}$$

holds for any $f \in L^2(\mathbb{R})$ and $\eta \in L^2(U)$ with $U \subset \mathbb{R}^2$ compact.

The original case that Kailath considered becomes somewhat trivial under this perspective. Indeed, if the spreading function is supported in a rectangle with width r and height $1/r$, then (9) already shows $\Phi_{\sqrt{r}g_r}$ is unitary from $L^2(U_r)$ to $L^2(\mathbb{R})$, since its adjoint, $V_{\sqrt{r}g_r}$ restricted to U_r, is essentially the corresponding Zak transform, which is unitary onto such a rectangle.

Hence we now consider a bit more complicated case where the channel is still underspread but the spreading support can not be included in a rectangle of area 1. Let U be a compact underspread Jordan measurable set, we include it in a $\sqrt{n} \times \sqrt{n}$ square where $n \in \mathbb{N}$ is large enough, and view the time-frequency plane as a torus with this $\sqrt{n} \times \sqrt{n}$ square being its fundamental domain. Under this setting without loss of generality we may assume that U is in the first sector of the time-frequency plane and take the square to be $[0, \sqrt{n}] \times [0, \sqrt{n}]$. The discretization procedure, proposed in [9] and presented with a different and simpler proof here, consists of three steps: rectification, vectorization and assembling the matrix.

Rectification:
We split the this square into a grid consists of cells with size $1/\sqrt{n} \times 1/\sqrt{n}$, so that the grid has $n \times n$ cells in total. We index each cell by a member in the additive group

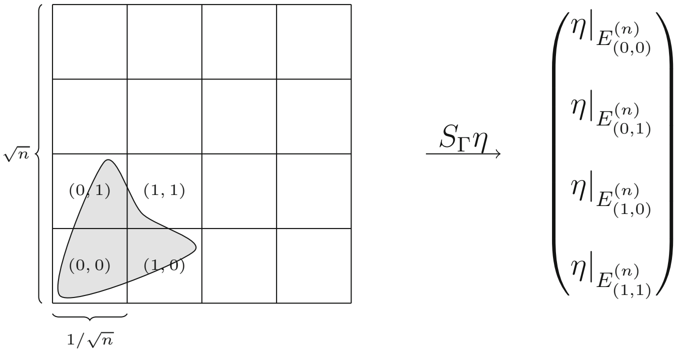

Fig. 3 Rectification and Vectorization

$\mathbb{Z}_n \times \mathbb{Z}_n$. The intersection cell of the j-th column from left to right and kth-row from bottom to top will be indexed as $E^{(n)}_{(j-1,k-1)}$, the superscript (n) indicates grid and cell sizes.

If $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$, we use the notation $E^{(n)}_\Lambda$ to denote the union of all cells indexed by Λ, i.e.,

$$E^{(n)}_\Lambda = \{E^{(n)}_{(j,k)} : (j,k) \in \Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n\}. \tag{10}$$

The full grid can thus be written as $E^{(n)}_{\mathbb{Z}_n \times \mathbb{Z}_n}$. In particular, the area of $E^{(n)}_\Lambda$ is $|\Lambda|/n$. We consider all cells that intersects U and set

$$\Gamma = \left\{(j,k) : \; E^{(n)}_{(j,k)} \bigcap U \neq \emptyset \right\}. \tag{11}$$

Vectorization:
Given $\Lambda \subset \mathbb{Z}_n \times \mathbb{Z}_n$, define the vectorization operator

$$S_\Lambda : L^2(E^{(n)}_\Lambda) \to L^2(E^{(n)}_{(0,0)})^{|\Lambda|},$$

so that if $(j,k) \in \Lambda$, then the (j,k)-th entry of $S_\Lambda \eta$ is η restricted to the cell $E^{(n)}_{(j,k)}$, i.e.,

$$(S_\Lambda \eta)_{(j,k)}(t,v) = \eta|_{E^{(n)}_{(j,k)}} = \eta\left(t + \frac{j}{r}, v + \frac{k}{r}\right), \tag{12}$$

where $(t,v) \in E^{(n)}_{(0,0)}$, and $r = \sqrt{n}$. The action of S_Λ is best described by the figure below (Fig. 3):

Assembling the matrix:
Take $\vec{c} \in \mathbb{C}^n$, for any $\eta \in L^2(U)$, we instead view it as an element in $L^2(E^{(n)}_\Gamma)$, then since

$$(V_{\vec{g}_c} f)(t,v) = \sum_{k=0}^{n-1} c_k \left\langle f, M_v T_{t+\frac{k}{r}} g_r \right\rangle = \sum_{k=0}^{n-1} c_k (V_{g_r} f)\left(t + \frac{k}{r}, v\right),$$

we obtain

$$(V_{\vec{g}_c} f)|_{E^{(n)}_{(0,0)}} = \sum_{k=0}^{n-1} c_k h_k = \langle \vec{h}, \bar{c} \rangle,$$

where $\bar{c}$ denotes the complex conjugate of $\vec{c}$, and

$$\vec{h} = S_{\mathbb{Z}_n \times \{0\}} V_{g_r} f. \tag{13}$$

Now if we move horizontally by one cell, with quasi-periodicity and $r^2 = n$ we get

$$\begin{aligned}(V_{\vec{g}_c} f)|_{E^{(n)}_{(1,0)}} &= (V_{\vec{g}_c} f)|_{E^{(n)}_{(0,0)}} \left(t + \frac{1}{r}, v\right) = \sum_{k=0}^{n-1} c_k (V_{g_r} f)|_{E^{(n)}_{(0,0)}} \left(t + \frac{k}{r} + \frac{1}{r}, v\right) \\ &= \langle \vec{h}, T\bar{c} \rangle, \end{aligned} \tag{14}$$

alternatively if we move vertically by one cell, then with similar reasoning we obtain

$$(V_{\vec{g}_c} f)|_{E^{(n)}_{(0,1)}} = \sum_{k=0}^{n-1} c_k (V_{g_r} f)|_{E^{(n)}_{(0,0)}} \left(t + \frac{k}{r}, v + \frac{1}{r}\right) = e^{-2\pi i \frac{t}{r}} \langle \vec{h}, M\bar{c} \rangle, \tag{15}$$

combing the above altogether we get the following conclusion:

Proposition 1 *Let $\vec{c} \in \mathbb{C}^n$, denote its complex conjugate as $\bar{c}$. Let U be a compact Jordan measurable set, let $r = \sqrt{n}$, and Γ be as defined in (11), then for any $\eta \in L^2(U)$, we have*

$$\Phi_{g_{\bar{c}}} \eta = \Phi_{g_r} S^{-1}_{\mathbb{Z}_n \times \{0\}} G_\Gamma(\bar{c}) D S_\Gamma \eta,$$

where D is a unitary diagonal scaling, and the ordering of columns in $G_\Gamma(\bar{c})$ is the same as the ordering of entries in S_Γ. Moreover, we also have

$$\frac{1}{\sqrt[4]{n}} \sigma_{\min}(G_\Gamma(\bar{c})) \|\eta\|_{L^2(U)} \leq \|\Phi_{g_{\bar{c}}} \eta\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt[4]{n}} \sigma_{\max}(G_\Gamma(\bar{c})) \|\eta\|_{L^2(U)},$$

where $\sigma_{\max}(G_\Gamma(\bar{c}))$ and $\sigma_{\min}(G_\Gamma(\bar{c}))$ are respectively the largest and the smallest singular values of of the Gabor matrix $G_\Gamma(\bar{c})$.

Proof With above derivations (14) and (15) we may decompose and vectorize $V_{g_{\bar{c}}} f$ on $E^{(n)}_\Gamma$ into

$$S_\Gamma V_{g_{\bar{c}}} f = D^* G^*_\Gamma(\bar{c}) S_{\mathbb{Z}_n \times \{0\}} V_{g_r} f,$$

where $*$ denotes the adjoint, and D^* is the unitary diagonal scaling that collects the exponential factor $e^{-2\pi i \frac{t}{r}}$ emerged in (15). The adjoint relation in (9) then gives the decomposition formula (It is easy to see that S_Γ and $S_{\mathbb{Z}\times\{0\}}$ are unitary, thus their adjoints and their inverses coincide).

The norm estimate follows by noticing that $S^{-1}_{\mathbb{Z}_n\times\{0\}}$, D, S_Γ are unitary, while the range of $S^{-1}_{\mathbb{Z}_n\times\{0\}}$ is $L^2([0,r]\times[0,1/r])$, on which Φ_{g_r} $(r=\sqrt{n})$ is, up to a scaling factor $\sqrt{r}$, also unitary. □

Consequently, injectivity of $\Phi_{g_{\vec{c}}}$, and in particular the upper and lower bound of $\Phi_{g_{\vec{c}}}$ depends solely on the Gabor matrix $G_\Gamma(\bar{c})$. Notice that it requires $|\Gamma| = n$ for $G_\Gamma(\bar{c})$ to be a square matrix, which means $E^{(n)}_\Gamma$ has area 1. On the other hand, for any compact underspread Jordan measurable set U we can always choose n large enough to cover it by such a set $E^{(n)}_\Gamma$.

This decomposition also complies with the definition of discretized channels in application. A discrete channel on $\mathbb{C}^{n\times n}$ is a weighted superposition of discrete translations and modulations, it takes the form of a linear combination $\sum_{(j,k)\in\Gamma} a_{jk} M^j T^k$ with $\Gamma \subset \mathbb{Z}_n \times \mathbb{Z}_n$. Therefore its response on an input $\vec{c}$ is simply $\sum_{(j,k)\in\Gamma} a_{jk} M^j T^k \vec{c}$, which can be viewed as the Gabor matrix $[M^j T^k \vec{c}]_{(j,k)\in\Gamma}$ multiplying the vector $\vec{a} = (a_{jk})^T_{(j,k)\in\Gamma}$.

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