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Universal Spectra in $G \times \mathbb{Z}_p$

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Abstract

Let G be an additive and finite Abelian group, and p a prime number that does not divide the order of G. We show that if G has the universal spectrum property, then so does $G \times \mathbb{Z}_p$. This is similar to and extends a previous result for cyclic groups using the same dilation trick but on non-cyclic groups as well. Inductively applying this statement on the known list of permissible G one can replace p with any square-free number that does not divide the order of G, and produce new tiling to spectral results in finite Abelian groups generated by at most two elements.

Keywords Fuglede conjecture · Universal spectrum · Tiling sets · Spectral sets

Mathematics Subject Classification 42A99

1 Introduction

Definition 1 A multiset is a collection of elements in which elements are allowed to be repeated. A set in the usual sense is then a special case of multisets in which every element has multiplicity 1.

In this article multisets will only be used after specifications. Let A, B be subsets of an additive Abelian group G, we use $A + B$ for the multiset formed by elements of form $a + b$ (computed within G) where a, b are enumerated from A, B respectively. We write $A \oplus B$ if $A + B$ is actually a set in the usual sense.

Definition 2 A subset A is said to be a tiling set in a finite additive Abelian group G, if there is another subset B of G such that $A \oplus B = G$ (i.e., $g \in G$ can be uniquely

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decomposed as $g = a + b$ with $a \in A$ and $b \in B$). B is then said to be a tiling complement of A, and $(A, \widehat{S})$ is called a tiling pair in G.

Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \quad \widehat{\mathbb{Z}_n} = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\right\},$$

respectively be the additive cyclic group of n elements and its dual. To better distinguish forms of elements in different sets, we will always use a hat symbol when talking about subsets in $\widehat{\mathbb{Z}_n}$, i.e., if $S \subseteq \mathbb{Z}_n$, then $\widehat{S} = \{s/n : s \in S\}$. To use notations consistently we agree that if $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is an additive and finite Abelian group, then we take $\widehat{G} = \widehat{\mathbb{Z}_{n_1}} \times \cdots \times \widehat{\mathbb{Z}_{n_k}}$.

If $x, y \in \mathbb{R}^d$, then we denote $x \cdot y$ as the Euclidean inner product of x and y.

Definition 3 A subset A is said to be a spectral set in a finite additive Abelian group G, if there is another subset $\widehat{S}$ of $\widehat{G}$ such that $\{e^{2\pi i s\cdot x}\}_{s\in\widehat{S}}$ is an orthogonal basis on $L^2(A)$ with respect to the counting measure. $\widehat{S}$ is then said to be a spectrum of A, and $(A, \widehat{S})$ is called a spectral pair in G.

Clearly $(A, \widehat{S})$ is a spectral pair if and only if $(S, \widehat{A})$ is also a spectral pair.

A famous and central problem concerning spectral sets and tiling sets is the Fuglede conjecture [6], which says that a set in $\mathbb{R}^d$ is a spectral set (in $\mathbb{R}^d$) if and only if it is a tiling set (in $\mathbb{R}^d$). Originally Fuglede considered only open connected sets with certain properties (Nykodym regions) and proved the statement for fundamental domains of lattices, he also showed that disks and triangles are not spectral. The conjecture has been disproved in both directions for $d \geq 3$, see e.g. [5, 14, 15, 25, 33]. For lower dimensions it remains open. Despite aforementioned negative results, there are still positive results for domains that naturally appear in applications, see e.g., [23].

An effective strategy for attacking this problem in $\mathbb{R}^d$ is to try to reduce it to finite Abelian groups. Those counterexamples mentioned above in $\mathbb{R}^d$ for $d \geq 3$ are constructed via complex Hadamard matrices which arise from different tiling and spectral sets in finite Abelian groups. Standard statements as in [25, Propositions 2.1 and 2.5] and [15, Theorems 4.1 and 4.2] indicate that positivity of the conjecture in $\mathbb{R}^d$ always imply positivity of the conjecture in finite Abelian groups generated by d elements. The other direction is only established partially, the case of $d = 1$ is elaborated in [3, Theorems 1.3 and 3.2] using rationality and periodicity results in [9, 13, 20], while for $d = 2$ periodicity is only known to a limited extent, see [1, 7, 8, 11, 26].

The result with most generality so far for the tiling to spectral direction in finite cyclic groups is to combine the Coven-Meyerowitz property [2] with associated Łaba spectra [16], latest progress on positive results of the Fuglede conjecture in cyclic groups are scattered in [12, 16, 18, 19, 21, 24, 28, 30]. Approaches for finite Abelian groups generated by two elements have been mostly combinatorial, see [4, 10, 29, 35].

In this paper we consider the universal spectrum property proposed in [21]:

Definition 4 If A is a tiling set in a finite and additive Abelian group G, and there is some $\widehat{S} \subseteq \widehat{G}$ such that $(B, \widehat{S})$ is a spectral pair for any tiling complement B of A, then $\widehat{S}$ is called a universal spectrum for tiling complements of A. If every tiling set in G has a universal spectrum for its tiling complements, then G is said to have the universal spectrum property.

It is shown in [21] that the strong Tijdeman property implies the universal spectrum property, and all cyclic "good" groups (a term used in the context of Hajós conjecture, see reference therein or perhaps expositions in [32, Chap. 4.2]) have these two properties. Upon considering all $n \in \mathbb{N}$, positivity of tiling to spectral is also equivalent to existence of universal spectra, see [5, 21]. Counterexamples (i.e., tiling sets without universal spectra) in Abelian groups generated by 3 or more elements can be found in [5, 15]. If G is cyclic, then Łaba spectra as constructed in [16] are actually universal.

The purpose of this article is to show that if G is an additive and finite Abelian group, and p is a prime number that does not divide the order of G, then the universal spectrum property can pass on from G to $G \times \mathbb{Z}_p$. Inductively applying this statement on the known list of permissible G one can replace p with any square-free number that does not divide the order of G, and produces new tiling to spectral results in finite Abelian groups generated by at most two elements.

2 Preliminaries

Let S be a subset of an additive Abelian group G, denote the difference set of S as

$$\Delta S = \{s - s' : \ s, s' \in S, \ s \neq s'\}.$$

If A, B are subsets of an additive Abelian group G, then clearly $A + B = A \oplus B$ if and only if the map

$$(a, b) \mapsto a + b, \quad a \in A, b \in B,$$

is injective. This map is not injective if and only if there exist distinct $a, a' \in A$ and $b, b' \in B$ with $a + b = a' + b'$, i.e., $a - a' = b' - b$, consequently we get

$$A + B = A \oplus B \quad \Leftrightarrow \quad \Delta A \cap \Delta B = \emptyset. \tag{1}$$

Let A be a subset of an additive and finite Abelian group G, and $s \in \widehat{G}$. Define

$$F_A(s) = \sum_{a \in A} e^{2\pi i a \cdot s}, \quad s \in \mathbb{R}^n.$$

Given a function f, denote

$$Z(f) = \left\{t : f(e^{2\pi i t}) = 0\right\}.$$

It is then easy to see that $(A, \widehat{S})$ is a spectral pair in G if and only if

$$\Delta\widehat{S} \subseteq Z(F_A), \quad |A| = |\widehat{S}|, \tag{2}$$

and (A, B) is a tiling pair in G if and only if

$$\Delta\widehat{G} \subseteq Z(F_A \cdot F_B), \quad |A| \cdot |B| = |G|. \tag{3}$$

A very special case of tiling pairs is when one component sits in a proper subgroup, then it has to tile the proper subgroup first, and allows an inductive analysis in the subsequent part (the lemma below is essentially [32, Lemma 2.4], with extra specifics on s_k):

Lemma 1 *Let $(A, B), (H, S)$ both be tiling pairs in an additive and finite Abelian group G, where H is a proper subgroup of G with index d, and $S = \{s_0, \ldots, s_{d-1}\}$ with $s_0 = 0$. For each $k \in \mathbb{Z}_d$, denote*

$$B^{(k)} = B \cap (s_k + H), \quad C^{(k)} = B^{(k)} - s_k.$$

If $A \subseteq H$, then $(A, C^{(k)})$ is a tiling pair in H for each $k \in \mathbb{Z}_d$.

Proof Each element in G can be written uniquely as $h + s$ for some $h \in H$ and $s \in S$. Applying the projection $h + s \mapsto s$ on the decomposition

$$G = A \oplus B = A \oplus \left(B^{(0)} \cup \cdots \cup B^{(d-1)}\right),$$

we obtain

$$S^{|H|} = \{0\}^{|A|} \oplus \left(\bigcup_{k \in \mathbb{Z}_d} \{s_k\}^{|B^{(k)}|}\right),$$

where notations such as $S^{|H|}$ mean the multiset formed by repeating every element of S precisely $|H|$ times. Comparing both sides it is easy to see that we must have

$$|B^{(0)}| = \cdots = |B^{(d-1)}| = \frac{|H|}{|A|}, \quad A \oplus B^{(k)} = s_k + H,$$

which is the desired result. □

A typical strategy to make use of Lemma 1 is the dilation result below, the cyclic case is given in [34, Theorem 1] while the non-cyclic case is later proved in [27, Proposition 3] (alternatively see [32, Theorem 3.17]).

Proposition 1 *Let G be an additive and finite Abelian group, and (A, B) a tiling pair in G. If $m \in \mathbb{N}$ is a number that is co-prime to $|A|$, then (mA, B) is still a tiling pair in G.*

3 Main Results

Theorem 1 *Let G be an additive and finite Abelian group, and p a prime number that does not divide $|G|$. If G has the universal spectrum property, then so does $G \times \mathbb{Z}_p$.*

Proof Let (A, B) be a non-trivial (i.e., both $|A|$ and $|B|$ are greater than 1) tiling pair in $G \times \mathbb{Z}_p$, since p does not divide $|G|$, one of $|A|$, $|B|$, say $|A|$ without loss of generality, is not divisible by p.

Let ψ be the projection from $G \times \mathbb{Z}_p$ to $G \times \{0\}$ with kernel H, i.e.,

$$\psi(g, j) = (g, 0).$$

Claim: $(\psi(A), B)$ is still a tiling pair in $G \times \mathbb{Z}_p$.

Indeed, since $p \nmid |A|$, (pA, B) remains a tiling pair in G by Proposition 1. Let p^{-1} be the multiplicative inverse of p modulo $|G|$, p^{-1} exists and is also co-prime to $|G|$ since both p, p^{-1} can be viewed as elements in the multiplicative group modulo $|G|$, consequently using Proposition 1 again we have $(p^{-1}pA, B)$ is a tiling pair in G, the claim thus follows since

$$\psi(A) = p^{-1}(pA).$$

Now for each $k \in \mathbb{Z}_p$ set

$$G^{(k)} = G \times \{0\} + (0, k), \quad B^{(k)} = B \cap G^{(k)}, \quad C^{(k)} = B^{(k)} - (0, k) = \psi(B^{(k)}).$$

By Lemma 1, $(\psi(A), B^{(0)})$ forms a tiling pair of $G \times \{0\}$, which has the universal spectrum property by assumption. Let $\widehat{S}_A \subset \widehat{G} \times \{0\}$ be a universal spectrum for all tiling complements of $B^{(0)}$ in $G \times \{0\}$, and similarly let $\widehat{S}_B \subset \widehat{G} \times \{0\}$ be a universal spectrum for all tiling complements of $\psi(A)$ in $G \times \{0\}$.

As $\widehat{S_A} \subseteq \widehat{G} \times \{0\}$, it is easy to see that

$$F_A(s_a) = F_{\psi(A)}(s_a) = 0, \tag{4}$$

holds for any $s_a \in \Delta\widehat{S_A}$, which implies that $(A, \widehat{S_A})$ is a spectral pair in $G \times \mathbb{Z}_p$, it is universal since by construction $\widehat{S}_A$ is independent of A.

By Lemma 1, for each $k \in \mathbb{Z}_p$, $(\psi(A), C^{(k)})$ is a tiling pair of $G \times \{0\}$, hence each $(C^{(k)}, \widehat{S}_B)$ is a spectral pair by universality of $\widehat{S}_B$. Repeating the same argument as in (4) we see that each $(B^{(k)}, \widehat{S}_B)$ is a spectral pair.

Let

$$H = \{0\} \times \mathbb{Z}_p,$$

we shall verify that $(B, \widehat{S_B} \oplus \widehat{H})$ is also a spectral pair:

By (1) it is clear that $\widehat{S_B} \oplus \widehat{H}$ is well defined, and thus

$$|B| = p|B^{(0)}| = p|\widehat{S_B}| = |\widehat{S_B} \oplus \widehat{H}|.$$

Take any pair of distinct elements $\hat{b}, \hat{b}' \in \widehat{B}$, if $\hat{b}, \hat{b}' \in \widehat{B^{(k)}}$ for some $k \in \mathbb{Z}_p$, then $\hat{b} - \hat{b}' \in \Delta\widehat{B^{(k)}} \subseteq Z(F_{S_B})$.

Alternatively if $\hat{b} \in \widehat{B^{(k)}}$ and $\hat{b}' \in \widehat{B^{(k')}}$ for distinct $k, k' \in \mathbb{Z}_p$, then by the definition of H we have

$$F_H(\hat{b} - \hat{b}') = F_H\left(\left(0, \frac{k-k'}{p}\right)\right) = \Phi_p(e^{2\pi(k-k')/p}) = 0.$$

where $\Phi_p(x) = \sum_{j=0}^{p-1} x^j$ is the p-the cyclotomic polynomial.

Therefore

$$\Delta\widehat{B} \subset Z(F_H \cdot F_{S_B}) = Z(F_{H \oplus S_B}),$$

which establishes spectrality by (2). The spectrum $\widehat{S_B} \oplus \widehat{H}$ is universal since by construction it is independent of B. □

Remark: A similar statement is available for the subcase of cyclic G in [17, Theorem 6.1, Corollary 6.2], using the dilation result for cyclic groups [34, Theorem 1], and the fact that Łaba spectra [16] of tiling sets with the Coven-Meyerowitz property [2] are universal. These arguments there goes back to [2] and actually also [21, Sect. 5]. It is also worth mentioning that as conjectured in [17, Sect. 9], all known approaches for classifying tiling pairs in cyclic G, including the Coven-Meyerowitz property (in particular, see the remark right after [2, Lemma 2.1]), quasi-periodicity (see perhaps expositions in [32, Chap. 5.2]), and Tijdeman properties (which has been disproved by counterexamples in [22, 31]), lead to the same intermediate statement that one component of a tiling pair shall be able to tile a smaller subset in G first.

Corollary 1 *Let $a, b, n \in \mathbb{N}$, and p, q, r distinct primes. If G is one of the following:*

$$\mathbb{Z}_{p^a q^b}, \quad \mathbb{Z}_{p^2 q^2 r^2}, \quad \mathbb{Z}_p \times \mathbb{Z}_{p^n}, \quad \mathbb{Z}_{pq} \times \mathbb{Z}_{pq},$$

and n_0 a square-free number that is co-prime to $|G|$, then any tiling set in $G \times \mathbb{Z}_{n_0}$ is also spectral, and has a universal spectrum for its tiling complements.

Proof It suffices to establish that each G possess the universal spectrum property, the rest then follows by successively applying Theorem 1 on each prime divisor of n_0.

It is shown in [2] (for $G = \mathbb{Z}_{p^n q^m}$) and [18, 19] (for $G = \mathbb{Z}_{p^2 q^2 r^2}$) that all tiling sets in G have Łaba spectra, which are universal.

Universalities in $\mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$ can be inferred directly from the construction in [4]. We give an outline below:

Let B be a fixed tiling set in $G = \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$, and consider an arbitrary tiling complement A of it, there are four cases:

- If $|B| = pq^2$, then $|A| = p$, according to [4, p. 13 Case 1], there is an order p element $\hat{u} \in \widehat{G}$ such that $\hat{u} \notin Z(F_B)$, and $(A, \langle\hat{u}\rangle)$ is a spectral pair in G for any tiling complement A of B (here the notation $\langle\hat{u}\rangle$ means the cyclic subgroup generated by $\hat{u}$), $\langle\hat{u}\rangle$ is universal since it depends only on B.

- If $|B| = q^2$, then $|A| = p^2$, according to [4, p. 14 Case 2], $\widehat{\mathbb{Z}_p} \times \widehat{\mathbb{Z}_p}$ is a spectrum of A, it is universal since in this case it is unique.
- If $|B| = pq$, then $|A| = pq$, according to [4, p. 14 Case 3], there exist $\hat{u}, \hat{v} \in \widehat{G} \setminus Z(F_B)$ with order p, q respectively, and $(A, \langle \hat{u}, \hat{v} \rangle)$ is a spectral pair in G for any tiling complement A of B (here the notation $\langle \hat{u}, \hat{v} \rangle$ means the subgroup generated by $\hat{u}$ and $\hat{v}$), the spectrum is universal since it does not depend on A.
- If $|B| = p$, then $|A| = pq^2$, according to [4, p. 13 Case 4], there is an order p element $\hat{u} \in \widehat{G}$ such that $\hat{u} \notin Z(F_B)$, and $\left(A, (\widehat{\mathbb{Z}_q} \times \widehat{\mathbb{Z}_q}) \oplus \langle \hat{u} \rangle\right)$ is a spectral pair in G for any tiling complement A of B, the spectrum is again universal since it is independent of A.

Finally to see universalities in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$, let B be a fixed tiling set in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$, and consider an arbitrary tiling complement A of it. We shall produce a spectrum $\widehat{S}$ of A that is also independent of A, then it is universal. Following the steps in [35], there are several cases:

- If $|B| = p^n$, then $|A| = p$. By [35, Lemma 2.7, Theorem 4.1], there is some $\hat{s} \in \widehat{\mathbb{Z}_p} \times \widehat{\mathbb{Z}_{p^n}} \setminus Z(F_B)$, and

$$\widehat{S} = \{k\hat{s} : k \in \mathbb{Z}_p\},$$

  is a spectrum of A, clearly $\widehat{S}$ depends only on B.
- If $|B| = p^b < p^n$, then $|A| = p^a$ where $a = n + 1 - b$. We partition the set $\widehat{X} = \{(0, 1), (0, p^{-1}), \ldots, (0, p^{-(n-1)})\} \subset \widehat{\mathbb{Z}_p} \times \widehat{\mathbb{Z}_{p^n}}$ into two parts:

$$\widehat{J} = Z(F_B) \cap \widehat{X}, \quad \widehat{I} = \widehat{X} \setminus \widehat{J}.$$

  Notice that by (3) this implies $\widehat{I} \subset Z(F_A)$.
  As pointed out by the first paragraph in the proof of [35, Theorem 4.2], it follows from [35, Lemma 3.2] that $|\widehat{J}|$ is either b or $b - 1$.

  - If $|\widehat{J}| = b - 1$, then $|\widehat{I}| = a$. By the second paragraph in the proof of [35, Theorem 4.2], the set

$$\widehat{S} = \left\{ \sum_{\hat{s} \in \widehat{I}} k\hat{s} : k \in \mathbb{Z}_p \right\},$$

    is a spectrum of A, and it is independent of A.
  - If $|\widehat{J}| = b$, then $|\widehat{I}| = a - 1$. As indicated in the proof of [35, Theorem 4.2] (from the third paragraph to the end of the proof), there are two further subcases: there exists some $\hat{d} \in \widehat{\mathbb{Z}_p} \times \{0\}$ and $\hat{h} \in \widehat{J}$ such that $\hat{d} + \hat{h} \notin Z(F_B)$ ([35, p. 9 case 2], so that by (3) $\hat{d} + \hat{h} \in Z(F_A)$ has to hold) and no such $\hat{d}, \hat{h}$ exist [35, p. 10 case 3], all other cases are ruled out by [35, p. 9 case 1]. In the

first case

$$\widehat{S} = \left\{ j(\hat{d} + \hat{h}) + \sum_{\hat{s} \in \widehat{I}} k\hat{s} : j, k \in \mathbb{Z}_p \right\},$$

is a spectrum of A, while in the second case

$$\widehat{S} = \left\{ (j/p, 0) + \sum_{\hat{s} \in \widehat{I}} k\hat{s} : j, k \in \mathbb{Z}_p \right\},$$

is a spectrum of A. In both situations $\widehat{S}$ is universal since it does not depend on A.

□

To the author's knowledge, concerning tiling to spectral in finite Abelian groups generated by at most two elements, this list of G produces the result with the most generality so far, in the sense that all known cases are covered in it and new cases are also discovered here using this list.

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References

1. Bhattacharya, S.: Periodicity and decidability of tilings of $\mathbb{Z}^2$. Am. J. Math. **142**, 255–266 (2020)
2. Coven, E., Meyerowitz, A.: Tiling the integers with translates of one finite set. J. Algebra **212**, 161–174 (1999)
3. Dutkay, D., Lai, C.: Some reduction of the spectral set conjecture on integers. Math. Proc. Camb. Philos. Soc. **156**(1), 123–135 (2014)
4. Fallon, T., Kiss, G., Somlai, G.: Spectral sets and tiles in $\mathbb{Z}_p^2 \times \mathbb{Z}_q^2$. J. Funct. Anal. **282**(12), 109472 (2022). https://doi.org/10.1016/j.jfa.2022.109472
5. Farkas, B., Matolcsi, M., Móra, P.: On Fuglede's conjecture and the existence of universal spectra. J. Fourier Anal. Appl. **12**(5), 483–494 (2006)
6. Fuglede, B.: Commuting self-adjoint partial differential operators and a group theoretic problem. J. Funct. Anal. **16**, 101–121 (1974)
7. Greenfeld, R., Tao, T.: The structure of translational tilings in $\mathbb{Z}^d$. Discrete Anal. **16**, 28 (2021)
8. Greenfeld, R., Tao, T.: A counterexample to the periodic tiling conjecture. Preprint. arXiv: 2211.15847 (2022)
9. Iosevich, A., Kolountzakis, M.: Periodicity of the spectrum in dimension one. Anal. PDE **6**(4), 819–827 (2013)
10. Iosevich, A., Mayeli, A., Pakianathan, J.: The Fuglede conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$. Anal. PDE **10**(4), 757–764 (2017)
11. Kenyon, R.: Rigidity of planar tilings. Invent. Math. **107**, 637–651 (1992)
12. Kiss, G., Malikiosis, R., Somlai, G., Vizer, M.: Fuglede's conjecture holds for cyclic groups of order $pqrs$. J. Fourier Anal. Appl. **28**, 79 (2022)
13. Kolountzakis, M., Lagarias, J.: Structure of tilings of the line by a function. Duke Math. J. **82**, 653–678 (1996)
14. Kolountzakis, M., Matolcsi, M.: Complex Hadamard matrices and the spectral set conjecture. Collect. Math. **57**, 281–291 (2006)
15. Kolountzakis, M., Matolcsi, M.: Tiles with no spectra. Forum Math. **18**(3), 519–528 (2006)

16. Łaba, I.: The spectral set conjecture and multiplicative properties of roots of polynomials. J. Lond. Math. Soc. **65**(3), 661–671 (2002)
17. Łaba, I., Londner, I.: Combinatorial and harmonic-analytic methods for integer tilings. Forum Math. Pi **10**(8), 1–46 (2022)
18. Łaba, I., Londner, I.: Splitting for integer tilings and the Coven-Meyerowitz tiling conditions. Preprint arXiv: 2207.11809 (2022)
19. Łaba, I., Londner: The Coven-Meyerowitz tiling conditions for 3 odd prime factors. Invent. Math. (2022)
20. Lagarias, J., Wang, Y.: Tiling the line with translates of one tile. Invent. Math. **124**, 341–365 (1996)
21. Lagarias, J., Wang, Y.: Spectral sets and factorizations of finite Abelian groups. J. Funct. Anal. **145**, 73–98 (1997)
22. Lagarias, J., Szabó, S.: Universal spectra and Tijdeman's conjecture on factorization of cyclic groups. J. Fourier Anal. Appl. **7**, 63–70 (2001)
23. Lev, N., Matolcsi, M.: The Fuglede conjecture for convex domains is true in all dimensions. Acta Math. **228**, 385–420 (2022)
24. Malikiosis, R.: On the structure of spectral and tiling subsets of cyclic groups. Forum Math. Sigma **10**, 1–42 (2022)
25. Matolcsi, M.: Fuglede's conjecture fails in dimension 4. Proc. Am. Math. Soc. **133**(10), 3021–3026 (2005)
26. McMullen, P.: Convex bodies which tile space by translations. Mathematika **27**, 113–121 (1980)
27. Sands, A.: Replacement of factors by subgroups in the factorization of Abelian groups. Bull. Lond. Math. Soc. **32**(3), 297–304 (2000)
28. Shi, R.: Fuglede's conjecture holds on cyclic groups $\mathbb{Z}_{pqr}$. Discrete Anal. **14** (2019). https://doi.org/10.19086/da.10570
29. Shi, R.: Equi-distribution on planes and spectral set conjecture on $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. J. Lond. Math. Soc. **102**(2), 1030–1046 (2020)
30. Somlai, G.: Spectral sets in $\mathbb{Z}_{p^2qr}$ tile. Discrete Anal. **5**, 10 (2023)
31. Szabó, S.: A type of factorization of finite Abelian groups. Discrete Math. **54**, 121–124 (1985)
32. Szabó, S., Sands, A.: Factoring Groups into Subsets. CRC Press, Boca Raton (2008)
33. Tao, T.: Fuglede's conjecture is false in 5 and higher dimensions. Math. Res. Lett. **11**(2–3), 251–258 (2004)
34. Tijdeman, R.: Decomposition of the integers as a direct sum of two subsets. Number Theory Séminaire de Théorie des Nombres de Paris 1992–3, pp. 261–276. Cambridge University Press, Cambridge (1995)
35. Zhang, T.: Fuglede's conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. SIAM J. Discrete Math. **37**(2), 1180–1197 (2023). https://doi.org/10.1137/22M1493598

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