Journal of Functional Analysis 286 (2024) 110402

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

On rationality of spectrums for spectral sets in $\mathbb{R}$

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ARTICLE INFO

Article history:
Received 22 October 2022
Accepted 23 February 2024
Available online 13 March 2024
Communicated by Stefanie Petermichl

MSC:
42A99

Keywords:
Fuglede conjecture
Spectral sets
Exponential basis
Orthonormal basis

ABSTRACT

Let $\Omega \subset \mathbb{R}$ be a compact measurable set of measure 1 and with null boundary measure. We show that if Ω is a spectral set, then it admits a rational spectrum. The proof relies on the periodicity of spectrums shown in [16], and adopts the technique in [28] for analyzing zeros of exponential sums as well as the technique in [16] that relates the spectrum to the tiling of $|\hat{\mathbf{1}}_\Omega|^2$. The key technical ingredient we contribute that eventually leads to the result is the periodicity of values from an exponential sum on a certain subgroup of $\mathbb{Z}$, which characterizes the torsion part of the spectrum. An immediate consequence of this result, together with periodicity and rationality results in [16,28], is the equivalence between the Fuglede conjecture in $\mathbb{R}$ and in $\mathbb{Z}_n$.

1. Introduction

Definition 1. We shall call $\Omega \subset \mathbb{R}$ a region, if it is compact, measurable, with zero boundary measure and positive interior measure.

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[1] Funded by the General Program for Natural Science Research in Jiangsu Higher Institutions. No. 21KJD110001.

https://doi.org/10.1016/j.jfa.2024.110402
0022-1236/

Definition 2. Let $\Omega \subset \mathbb{R}$ be a region, it is said to be spectral if there is a countable set $\Lambda \subset \mathbb{R}$ such that the exponential system $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\Omega)$, Λ is then called the spectrum of Ω.

Definition 3. Let $\Omega \subset \mathbb{R}$ be a region, it is said to be tiling if there is a countable set $\Gamma \subset \mathbb{R}$ such that

$$\bigcup_{\gamma\in\Gamma} \gamma + \Omega = \mathbb{R},$$

holds up to the difference of a null set and the union is disjoint. Γ is then called the tiling complement of Ω. We may also say that Ω tiles $\mathbb{R}$ by Γ.

A famous problem concerning spectral sets and tiling sets is the Fuglede conjecture [10], which says that a set in $\mathbb{R}^d$ is spectral (in $\mathbb{R}^d$) if and only if it is tiling (in $\mathbb{R}^d$). Originally Fuglede considered only open connected sets with certain properties (Nykodym regions) and proved the statement for fundamental domains of lattices, he also showed that disks and triangles are not spectral. The conjecture has been disproved in both directions for $d \geq 3$, see e.g. [9,21,22,36,44]. For lower dimensions it remains open.

With many nice properties, exponential orthogonal bases are not only core objects in Fourier analysis but also important tools in application areas such as samplings and (in general) signal processing. The search for exponential orthogonal bases on different domains has been a central and intriguing topic. Despite aforementioned negative results, there are still positive results for domains that naturally appear in applications, see e.g., [17,23,33].

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the additive cyclic group of n elements, an effective strategy for this problem is to try to reduce it to finite Abelian groups. For example, those counterexamples mentioned above for $d \geq 3$ are constructed via complex Hadamard matrices which arise from different tiling and spectral sets in $\mathbb{Z}_n^d$. Standard arguments as in [36, Proposition 2.1, Proposition 2.5] and [22, Theorem 4.1, Theorem 4.2] indicate that positivity of the conjecture in $\mathbb{R}^d$ always imply positivity of the conjecture in finite Abelian groups generated by d elements. The opposite direction is only established partially for $d = 1$ in [8, Theorem 1.3, Theorem 3.2] using rationality and periodicity results in [16,20,28].

To be more precise, for $d = 1$, it was shown in [20,28] that normalized (i.e., containing 0) tiling complements of unit measure regions are periodic and rational, which builds the equivalence of the tiling to spectral direction between $\mathbb{R}$ and $\mathbb{Z}_n$ [8, Theorem 1.3, Theorem 3.2]. On the other hand, [16] showed that spectrums for bounded measurable sets with measure 1 must be periodic, but rationality of these spectrums is still missing (several partial results are available in e.g., [4,8,23,24]). Thus spectral to tiling in $\mathbb{R}$ implies spectral to tiling in $\mathbb{Z}_n$ but the converse statement is still to be established.

For cyclic groups, Hajós and Sands classified all “good” groups among $\mathbb{Z}_n$ in which all decompositions are periodic, and further conjectured that all decomposition of finite cyclic groups are quasi-periodic, see [5,15,37,38] or expository parts in e.g., [41,42]. Using Tijdeman properties ([45]) and the concept of universal spectrums (see also [9]), it was further shown in [29] that tiling to spectral holds for all cyclic "good" groups. Another effective approach is to use the Coven-Meyerowitz property ([6]) combined with associated Łaba spectrums ([24]), a list of latest results using this method can be found in [24–27]. For "good" cyclic groups, spectral to tiling also holds, see [19,34,35,39,40] and references therein, techniques classifying vanishing sums of roots of unity in [30] are also used there.

The purpose of this article is to show that if Ω is a spectral region of measure 1, then it has a rational spectrum. As a result, the Fuglede conjecture for regions in $\mathbb{R}$ is now equivalent to the Fuglede conjecture in $\mathbb{Z}_n$ (for all $n \in \mathbb{N}$).

The next section collects necessary facts and tools from the literature, then follows the section that further develops relevant techniques, the main result is presented in the last section.

2. Preliminaries

2.1. Characterization of spectrums

Results from this subsection are repeated from [16], with some extra subsidiary remarks.

Denote $\mathbf{1}_\Omega$ as the characteristic function on Ω, and $\hat{f}$ as the Fourier transform of a function f. We adopt the following form of the Fourier transform on the Schwartz class:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx,$$

and extend it to tempered distributions by duality.

Given a function $f : \mathbb{R} \to \mathbb{C}$, we shall denote its zero set by $Z(f)$, i.e.,

$$Z(f) = \{x \in \mathbb{R} : f(x) = 0\}.$$

For a countable set $\Lambda \subset \mathbb{R}$, we denote its difference set (or gap set) by

$$\Lambda - \Lambda = \{\lambda - \lambda' : \lambda \neq \lambda', \ \lambda, \lambda' \in \Lambda\}.$$

In the lemma below, (1) is well known, while (2) is taken from [16] (who further attributes it to [10]).

Lemma 1 *([16]). Given a region $\Omega \subset \mathbb{R}$ with measure m, then Λ is a spectrum of Ω if and only if*

$$\Lambda - \Lambda \subseteq Z(\hat{\mathbf{1}}_\Omega), \tag{1}$$

and

$$\sum_{\lambda\in\Lambda} |\hat{f}(\xi-\lambda)|^2 = m\cdot \|f\|_{L^2(\Omega)}, \tag{2}$$

holds for all $\xi \in \mathbb{R}$ *and* $f \in L^2(\Omega)$.

Proof. Straightforward computation shows that

$$\langle e^{2\pi i\lambda x}, e^{2\pi i\lambda' x}\rangle = \langle \mathbf{1}_\Omega, e^{2\pi i(\lambda'-\lambda)x}\rangle = \hat{\mathbf{1}}_\Omega(\lambda'-\lambda),$$

thus orthogonality is equivalent to (1). Similarly we have

$$\frac{1}{m}\sum_{\lambda\in\Lambda} |\hat{f}(\xi-\lambda)|^2 = \sum_{\lambda\in\Lambda} |\langle f, \frac{1}{\sqrt{m}} e^{2\pi i(\xi-\lambda)x}\rangle|^2,$$

hence both completeness and unity of the system $m^{-1/2}\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ are equivalent to (2). □

Definition 4. A countable set $\Lambda \subset \mathbb{R}$ is said to be periodic if there exists a positive number t (i.e., a period) such that

$$t + \Lambda = \Lambda,$$

it is said to be rational if actually $\Lambda \subseteq \mathbb{Q}$, and uniformly discrete if there is some constant $d > 0$ such that

$$|\lambda - \lambda'| \geq d,$$

holds for all distinct $\lambda, \lambda' \in \Lambda$.

Clearly spectrums must be uniformly discrete, the simplest way to see it is by using (1): Since Ω is compact, the Paley-Wiener theorem shows that $\hat{\mathbf{1}}_\Omega$ is analytic, thus there are no accumulation points in its zero set. In sampling theory, various notions of densities (e.g., Beurling density [2], uniform density [7], see [1, Chapter 1] and [47] for details) are also used to characterize the asymptotic average number of points in Λ for exponential frames $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$. A famous result is by Landau [31] on the necessary lower density, where sufficient conditions are available in e.g., [7,18]. Density results for $Z(\hat{\mathbf{1}}_\Omega) \cap \mathbb{Z}$ for certain region $\Omega \subset \mathbb{R}$ can be found in [28, Section 3], and a partial statement for regions in $\mathbb{R}^d$ is given in [29, Theorem 2.2]. The structure of spectrums is actually very rigid:

Lemma 2 *([16]). Suppose that* $\Omega \subset \mathbb{R}$ *is a region of measure* 1*, if* Ω *is spectral with spectrum* Λ*, then* Λ *is uniformly discrete and periodic, moreover, any period of* Λ *must be an integer.*

Proof. Details from the original proof are too long to be fully reproduced here, see [16, Theorem 1]. □

2.2. Zeros of exponential sums

Results from this subsection are essentially repeated from the main content of [28, Section 4], with some slight rephrasing to accommodate the purpose of this paper, because of this a proof following the same approach is also given.

For a finite set $S \subset \mathbb{R}$, let us denote

$$F_S(x) = \sum_{s \in S} e^{2\pi i s x}.$$

Lemma 3. *[28] Let $S \subset \mathbb{R}$ be a finite set, and set*

$$r = |S|.$$

Let $S_{\mathbb{Q}}$ denote the subset of rationals in S, and $S_{\mathbb{Q}}^c$ the subset of irrationals in S. Let

$$Z(F_S) = X \cup Y,$$

where X is the union of all periodic subsets of $Z(F_S)$ with rational periods, and Y is disjoint from X, then we have

(i)

$$X \subseteq Z(F_{S_{\mathbb{Q}}}) \cap Z(F_{S_{\mathbb{Q}}^c}).$$

(ii) Y contains no arithmetic progressions that are of length longer than $r-1$ and with rational common differences.

Proof. We shall first construct the set X. Consider the equivalence relation

$$s \sim s' \quad \Leftrightarrow \quad s - s' \in \mathbb{Q},$$

which creates a partition on S:

$$S = S_0 \cup S_1 \cup \ldots \cup S_n, \tag{3}$$

where $S_0 \subset \mathbb{Q}$ and $S_1, \ldots, S_n$ are subsets of irrationals, each set is an equivalence class under this equivalence relation.

For each $k \in \{0, \ldots, n\}$, let $r_k = |S_k|$, then we can write each S_k as

$$S_k = s_k + \left\{ \frac{p_{k,1}}{q_{k,1}}, \ldots, \frac{p_{k,r_k}}{q_{k,r_k}} \right\},$$

where $s_0 \in \mathbb{Q}$, $s_1, \ldots, s_n$ are irrationals, and $p_{k,1}, \ldots, p_{k,r_k}$ as well as $q_{k,1}, \ldots, q_{k,r_k}$ are natural numbers. Set

$$N_k = \operatorname{lcm}(q_{k,1}, \ldots, q_{k,r_k}),$$

where lcm is the notion for least common multipliers, then clearly $Z(F_{S_k})$ is N_k periodic if it is not empty.

Now we claim that

$$X = Z(F_{S_0}) \cap \ldots \cap Z(F_{S_n}).$$

By this construction, X would be periodic with period

$$N = \operatorname{lcm}(N_0, N_1, \ldots, N_n),$$

as long as it is not empty. Therefore it suffices to establish (ii), i.e., to show that

$$Y = Z(F_S) \setminus X = Z(F_S) \setminus \left(Z(F_{S_0}) \cap \ldots \cap Z(F_{S_n})\right),$$

contains no more than r consecutive elements from any arithmetic progression with rational common differences, which will then immediately imply both (i) and (ii).

Assume the contrary of (ii), suppose that $y, y+d, \ldots, y+(r-1)d \in Y$ for some $d \in \mathbb{Q} \setminus \{0\}$ and $y \in \mathbb{R}$, then we have

$$0 = F_S(y+jd) = \sum_{s \in S} e^{2\pi i s(y+jd)}, \quad j = 0, 1, \ldots, r-1. \tag{4}$$

Consider now a new equivalence relation

$$s \sim s' \quad \Leftrightarrow \quad e^{2\pi i s d} = e^{2\pi i s' d},$$

which in turn induces a new partition on S:

$$S = S'_1 \cup \ldots \cup S'_m, \tag{5}$$

where each of $S'_1, \ldots, S'_m$ is an equivalence class under this new equivalence relation. Pick any $s'_k \in S'_k$ ($k = 1, \ldots, m$), and set

$$z_k = e^{2\pi i s'_k d},$$

then (4) becomes

$$0 = F_S(y+jd) = \sum_{k=1}^{m} z_k^j F_{S'_k}(y), \quad j = 0, 1, \ldots, r-1,$$

which can be viewed as a linear system by applying an $r \times m$ sub-matrix from a Vandermonde matrix to the vector $\left(F_{S_1'}(y), \ldots, F_{S_m'}(y)\right)^T$. As $r \geq m$ and $z_1, \ldots, z_m$ are distinct, the matrix has full rank and thus the solution is trivial, i.e.,

$$F_{S_1'}(y) = \ldots = F_{S_m'}(y) = 0.$$

Notice that the partition (5) is finer than the partition (3) since $d \in \mathbb{Q}$, hence the above actually implies that

$$F_{S_0}(y) = \ldots = F_{S_n}(y) = 0,$$

i.e., $y \in X$, which is a contradiction since we have assumed that $X \cap Y = \emptyset$ and $y \in Y$. $\square$

It is shown in [28] using [11,12,43] and [32,46] that $Y \cap \mathbb{Z}$ is of zero density and in fact finite.

3. Technical preparations

In this part we will better describe the torsion part S of a spectrum $T\mathbb{Z} + S$ by looking into properties of F_S.

Denote δ_t as the Dirac distribution at t. Given a function $f : \mathbb{R} \to \mathbb{C}$, we write

$$\tilde{f}(x) = f(-x),$$

and let $*$ be the convolution operator, i.e.,

$$(f * f)(x) = \int_{\mathbb{R}} f(t) f(x - t) dt.$$

For a region $\Omega \subset \mathbb{R}$, denote its difference set by

$$\Omega - \Omega = \{x - x' : \; x, x' \in \Omega, \; x \neq x'\}.$$

It is then easy to see that $(\Omega - \Omega) \cup \{0\}$ is the support of $\mathbf{1}_\Omega * \tilde{\mathbf{1}}_\Omega$.

Lemma 4. *Let $\Omega \subset \mathbb{R}$ be a region of measure 1, if Ω is a spectral set with spectrum*

$$\Lambda = T\mathbb{Z} + S,$$

for some $T \in \mathbb{N}$ and some finite set $S \subset [0, T)$, and

$$E = (\Omega - \Omega) \cap T^{-1}\mathbb{Z}, \tag{6}$$

then we have

$$E \subseteq Z(F_S).$$

Proof. With (2) in Lemma 1 we have for any $f \in L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} = 1$ that

$$\sum_{\lambda\in\Lambda} |\hat{f}(\xi - \lambda)|^2 = 1.$$

Applying the inverse Fourier transform to both sides of the equation we get

$$\left(f * \tilde{f}\right)(x) \cdot \sum_{\lambda\in\Lambda} e^{2\pi i\lambda x} = \delta_0, \tag{7}$$

where the equality holds in the sense of tempered distributions.

Applying the distributional Poisson summation formula on (7) we obtain

$$\left(f * \tilde{f}\right)(x) \cdot F_S(x) \cdot \frac{1}{T} \sum_{k\in\mathbb{Z}} \delta_{\frac{k}{T}} = \delta_0. \tag{8}$$

Now taking $\mathbf{1}_\Omega$ as f, then for (8) to hold we necessarily have $E \subseteq Z(F_S)$. □

Remark. Consider $H, S \subset \mathbb{Z}_n$, $\Omega = [0,1] + H$ and $\Lambda = \mathbb{Z} + n^{-1}S$, then Λ is the spectrum of Ω if and only if S is the spectrum of H (see Definition 6) in $\mathbb{Z}_n$. For the setting in Z_n it is easy to see that S is the spectrum for H if and only if

$$H - H \subseteq Z(F_{n^{-1}S}),$$

thus Lemma 4 is simply the continuous analog of this characterization.

Lemma 5. *Let $\Omega \subset \mathbb{R}$ be a region of measure 1, suppose that Ω is spectral with spectrum*

$$\Lambda = T\mathbb{Z} + S,$$

for some $T \in \mathbb{N}$ and some finite set $S \subset [0, T)$. Assume further that $0 \in \Lambda$. Denote

$$R = \max\{t - t' : \ t, t' \in T^{-1}\mathbb{Z} \cap \Omega\}, \tag{9}$$

then F_S is $R + T^{-1}$ periodic on $T^{-1}\mathbb{Z}$, i.e.,

$$F_S(t + R + T^{-1}) = F_S(t),$$

holds for any $t \in T^{-1}\mathbb{Z}$.

Proof. Without loss of generality let us assume that

$$\Omega \subseteq [0, R + T^{-1}) \text{ and } 0 \in \Omega. \tag{10}$$

This assumption is legitimate since Ω is compact and shifting does not change spectrums.

Apparently $\Lambda \setminus \{0\}$ is a subset of $\Lambda - \Lambda$, and recall that $\hat{\mathbf{1}}_\Omega(0)$ equals the measure of Ω, which is 1, thus together with (1) we have

$$\hat{\mathbf{1}}_\Omega \cdot \sum_{\lambda \in \Lambda} \delta_\lambda = \hat{\mathbf{1}}_\Omega(0)\delta_0 = \delta_0,$$

where the equality holds in the sense of tempered distributions. Applying the inverse Fourier transform to both sides of the equation we obtain

$$\mathbf{1}_\Omega(x) * \sum_{\lambda \in \Lambda} e^{2\pi i \lambda x} = 1.$$

With the assumption that $\Lambda = T\mathbb{Z}+S$ and the distributional Poisson summation formula we get

$$\mathbf{1}_\Omega(x) * (F_S(x) \cdot \frac{1}{T} \sum_{k \in \mathbb{Z}} \delta_{\frac{k}{T}}) = 1, \tag{11}$$

which can be further written as

$$\sum_{k \in \mathbb{Z}} F_S(\frac{k}{T}) \mathbf{1}_\Omega(x - \frac{k}{T}) = T.$$

Consider now

$$x_0 = R + \frac{k_0}{T},$$

and

$$x_0' = R + \frac{k_0+1}{T},$$

for some arbitrarily chosen $k_0 \in \mathbb{Z}$. By (10) and (11) we have

$$T = \sum_{k \in \mathbb{Z}} F_S(\frac{k}{T}) \mathbf{1}_\Omega(x_0 - \frac{k}{T}) = \sum_{k=k_0}^{k_0+RT} F_S(\frac{k}{T}) \mathbf{1}_\Omega(R + \frac{k_0}{T} - \frac{k}{T}) = \sum_{k=k_0}^{k_0+RT} F_S(\frac{k}{T}),$$

and

$$\begin{aligned} T = \sum_{k \in \mathbb{Z}} F_S(\frac{k}{T}) \mathbf{1}_\Omega(x_0' - \frac{k}{T}) &= \sum_{k=k_0+1}^{k_0+RT+1} F_S(\frac{k}{T}) \mathbf{1}_\Omega(R + \frac{k_0+1}{T} - \frac{k}{T}) \\ &= \sum_{k=k_0+1}^{k_0+RT+1} F_S(\frac{k}{T}). \end{aligned}$$

Subtracting these two equations we get

$$0 = F_S(\frac{k_0 + RT + 1}{T}) - F_S(\frac{k_0}{T}),$$

i.e.,

$$F_S(\frac{k_0}{T} + R + \frac{1}{T}) = F_S(\frac{k_0}{T}),$$

which shows that F_S is $R + T^{-1}$ periodic on $T^{-1}\mathbb{Z}$ since k_0 is arbitrary. $\square$

4. Rationality of spectrums

The strategy is essentially the same as in [28], some arguments are adjusted to adapt to this problem. The periodicity in Lemma 5 plays a vital role here.

Theorem 1. *Let $\Omega \subset \mathbb{R}$ be a region of measure 1, if Ω is spectral with spectrum Λ, and $0 \in \Lambda$, then Λ is rational.*

Proof. By Lemma 2, we may write Λ as

$$\Lambda = T\mathbb{Z} + S,$$

where $T \in \mathbb{N}$, and $S \subset [0, T)$ is a finite set. Let E be as defined in (6), Lemma 4 then shows that

$$E \subset Z(F_S).$$

Now by Lemma 5, F_S is periodic on $T^{-1}\mathbb{Z}$ with period $R+T^{-1}$ where R is as defined in (9), thus if we denote

$$K = (R + T^{-1})\mathbb{Z} + E,$$

then this periodic set K is included in $Z(F_S)$. Now let

$$S_{\mathbb{Q}} = S \cap \mathbb{Q}, \quad \Lambda_{\mathbb{Q}} = \Lambda \cap \mathbb{Q} = T\mathbb{Z} + S_{\mathbb{Q}},$$

then $S_{\mathbb{Q}}$ is non-empty since $0 \in S$.

Observe that (1) in Lemma 1 is satisfied on $\Lambda_{\mathbb{Q}}$ as obviously we have

$$\Lambda_{\mathbb{Q}} - \Lambda_{\mathbb{Q}} \subseteq \Lambda - \Lambda \subseteq Z(\hat{\mathbf{1}}_\Omega).$$

On the other hand, by Lemma 3, $F_{S_{\mathbb{Q}}}$ and F_S share the same set of rational periodic zeros, thus we will also have

$$K \subseteq Z(F_{S_{\mathbb{Q}}}). \tag{12}$$

Consequently (2) in Lemma 1 is still satisfied if we replace Λ with $\Lambda_{\mathbb{Q}}$, since (7) and (8) will still hold due to (12) if S is replaced by $S_{\mathbb{Q}}$.

Therefore by Lemma 1, $\Lambda_{\mathbb{Q}}$ is also a spectrum of Ω, which is a contradiction if $S_{\mathbb{Q}}$ is only a proper subset of S. Hence we can conclude that

$$S = S_{\mathbb{Q}}. \quad \square$$

An immediate consequence of Theorem 1, combined with periodicity and rationality results in [16,28], is the equivalence between the Fuglede conjecture for regions in $\mathbb{R}$ and for $\mathbb{Z}_n$ (for all $n \in \mathbb{N}$).

It is most convenient to introduce tiling sets and spectral sets in $\mathbb{Z}_n$ through the Fourier matrix. The $n \times n$ Fourier matrix W is defined as the matrix whose rows and columns are both indexed by $\mathbb{Z}_n$ and whose jk-th entry is simply $n^{-1/2}e^{2\pi ijk/n}$ for $j, k \in \mathbb{Z}_n$ (with 0 identified as n). If $H, S \subseteq \mathbb{Z}_n$, then we shall denote $W_{H,S}$ as the submatrix in W composed by taking row indices from H and column indices from S.

Definition 5. A subset H is said to tile $\mathbb{Z}_n$ by H' (or to complement H' in $\mathbb{Z}_n$, or simply to be tiling in $\mathbb{Z}_n$), if there exists another subset H' with $H \times H' = \mathbb{Z}_n$, so that each element $g \in \mathbb{Z}_n$ can be uniquely decomposed as $g = h + h'$ with $h \in H$ and $h' \in H'$.

Definition 6. A subset H is said to be spectral in $\mathbb{Z}_n$, if there exists another subset S so that the submatrix $W_{H,S}$ is orthogonal. S is then called the spectrum of H.

Definition 7. A complex Hadamard matrix is an orthogonal complex matrix in which all entries have same moduli.

Since all entries in W have same moduli, it is also equivalent to say that H is spectral with spectrum S in $\mathbb{Z}_n$ if $W_{H,S}$ is a complex Hadamard submatrix in W.

The Fuglede conjecture in $\mathbb{Z}_n$ then says that a subset is spectral in $\mathbb{Z}_n$ if and only if it is tiling in $\mathbb{Z}_n$.

Corollary 1. *The Fuglede conjecture for regions in $\mathbb{R}$ is true if and only if the Fuglede conjecture for $\mathbb{Z}_n$ is true for all $n \in \mathbb{N}$. To be more precise, tiling to spectral in $\mathbb{R}$ is true if and only if it is true in $\mathbb{Z}_n$ for all $n \in \mathbb{N}$, and similarly spectral to tiling in $\mathbb{R}$ is true if and only if it is true in $\mathbb{Z}_n$ for all $n \in \mathbb{N}$.*

Proof. See [8, Theorem 1.3, Theorem 3.2].

The proof relies on a bunch of technical steps which are not to be repeated here. The essential part is structure theorems established in [28] for tiling sets and [16] for spectral sets, which says a tiling (resp. spectral) region Ω of unit measure can be written as $\bigcup_k B_k + s_k$ where $\bigcup_k B_k$ is a partition of the interval $[0, n^{-1})$ and s_k are elements from

$n^{-1}\mathbb{Z}_n$ for some $n \in \mathbb{N}$. These structure theorems lead to correspondences between Ω and the row index set H of submatrices in W. $\square$

It is also worth mentioning that the Fuglede conjecture in $\mathbb{Z}_n$ and in $\mathbb{R}$ is also equivalent to the so called universal tiling (resp. spectrum) conjecture which says that sets with the same tiling complement (resp. spectrum) also share the same spectrum (resp. tiling complement). The concept of either a universal tiling set or a universal spectral set are easy to grasp by looking at Hadamard submatrices of the Fourier matrix. These conjectures are proposed in [29] and [9], they are shown (also in [9]) to be false in $\mathbb{R}^3$ and higher dimensions, but are still open in $\mathbb{R}$ and $\mathbb{R}^2$.

The case on $\mathbb{R}^d$ for $d \geq 2$ is much more complicated, tiling complements or spectrums need not be periodic or rational in general even for nice domains. For example, $\{(a, \sqrt{a}\pi + b)\}_{a,b\in\mathbb{Z}}$ is a tiling complement of the unit square in $\mathbb{R}^2$, but it is neither rational nor periodic. That whether any tiling region in $\mathbb{R}^d$ would admit at least one periodic tiling complement (to be precise, a d-periodic tiling complement) is known as the periodic tiling conjecture (proposed also in [28]), and has recently been disproved for $d \geq 3$ [14], on $\mathbb{R}^2$ it remains open but on $\mathbb{Z}^2$ it has actually been established lately [3,13]. Thus there are still large gaps to extend Corollary 1 to $d = 2$, while for $d \geq 3$ it is possibly false (at least the method in the proof of Corollary 1 can no longer be used).

Data availability

No data was used for the research described in the article.

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