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QUASI-PERIODICITY OF $\mathbb{Z}_{p^a n_0}$

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Abstract. Let p^a be a prime power and n_0 a square-free number. We prove that any complementing pair in a cyclic group of order $p^a n_0$ is quasi-periodic, with one component decomposable by the the subgroup of order p. The proof is by induction and reduction since the presence of the square-free factor n_0 allows us to perform a Tijdeman decomposition. We also give an explicit example to show that $\mathbb{Z}_{72}$ is the smallest cyclic group that fails to have the strong Tijdeman property.

1. Introduction

Definition 1. A multiset is a collection of elements in which elements are allowed to be repeated. A simple set (i.e., a set in the usual sense) is then a special case of multisets in which every element has multiplicity 1.

The operation of taking subsets, unions, and intersections in this article are used only on simple sets and yield only simple sets. If A is a simple set and $k \in \mathbb{N}$, then we use $A^{\otimes k}$ to indicate the multiset in which every element of A is repeated k times.

Definition 2. Let A, B be subsets of an additive Abelian group G. We use $A + B$ for the multiset formed by elements of form $a + b$ (computed within G) where a, b are enumerated from A, B respectively. We write $A \oplus B$ if $A + B$ is actually a simple set.

If $g \in G$ and $A \subseteq G$, then we use the notation $g + A$ for the simple set $\{g + a : a \in A\}$ where $g + a$ is computed within G.

Definition 3. A subset A is said to tile an additive Abelian group G by B (or to complement B in G, or simply to be tiling in G), if there is some $B \subseteq G$ with $A \oplus B = G$, so that each element $g \in G$ can be uniquely decomposed as $g = a + b$ with $a \in A$ and $b \in B$. In such cases we shall call (A, B) a complementing pairing in G.

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Definition 4. A subset A of an additive Abelian group G is said to be periodic, if $g+A=A$ holds for some non-trivial element $g\in G$. Equivalently A is periodic if and only if $A=H\oplus A'$ for some subset $A'\subset A$ and some subgroup $H\lhd G$ that contains g (in particular, A' is allowed to be trivial, which means that A is a subgroup). A complementing pair (A,B) in G is called periodic if at least one of A,B is periodic. If all complementing pairs in G are periodic, then G is called a "good" group.

The study of such decompositions is initiated by Hajós [4] in order to solve a problem posed by Minkowski. Based also on works of de Brujin [1] and Rédei [7], a complete list of "good" groups is given by Sands [8,9] (alternatively see expository parts in e.g., [13,14,20]), namely

$$\begin{aligned}
\text{Rank 1}&: (p,q,r,s),\ (p^2,q,r),\ (p^a,q),\ (p^2,q^2),\\
\text{Rank 2}&: (3^2,3),\ (2^a,2),\ (2^2,2^2),\ (p,p),\\
\text{Other}&: (p^3,2,2),\ (p,2^2,2),\ (p,3,3),\ (p,q,2,2),\ (p^2,2,2,2),\ (p,2,2,2,2),
\end{aligned}$$

where $a\in\mathbb{N}$, p, q, r, s are distinct prime numbers, notions such as (p,q,r,s) means $\mathbb{Z}_p\times\mathbb{Z}_q\times\mathbb{Z}_r\times\mathbb{Z}_s$, where

$$\mathbb{Z}_n=\left\{0,1,\dots,n-1\right\},$$

is the additive cyclic group of n elements, and p,q are allowed to be 2 and 3 in cases of e.g., $(p,2^2,2)$, $(p,3,3)$, $(p,q,2,2)$, etc.

Definition 5. A complementing pair (A,B) in an additive Abelian group G is said to be quasi-periodic, if at least one of A,B, say B, can be partitioned into m equal-sized subsets $B^{(0)},\dots,B^{(m-1)}$, and there is an order m subgroup $H=\{h_0,\dots,h_{m-1}\}$ in G with $h_0=0$ so that

$$A\oplus B^{(k)}=h_k+A\oplus B^{(0)},$$

holds for each $k\in\{0,\dots,m-1\}$. In such case we shall call B decomposable by H in this complementing pair. G is said to be quasi-periodic if all complementing pairs in it are quasi-periodic.

Example 1 [1]. A decomposition that is quasi-periodic but not periodic:

$$\begin{aligned}
\mathbb{Z}_{72}&=\{0,8,16,18,26,34\}\oplus\{0,5,6,9,12,29,33,36,42,48,53,57\}\\
&=\{0,36\}\oplus\{0,8,16,18,26,34\}\oplus\{0,6,9,12,33,57\}.
\end{aligned}$$

Conjecture 1 (Hajós). *All finite cyclic groups are quasi-periodic.*

Definition 5 is from [3], originally periodic sets instead of subgroups were used to define quasi-periodicity. These two definitions are equivalent, see e.g.

[19, Chapter 5.2] or [12]. Initially Hajós made the above conjecture for all finite Abelian groups [4], but a counterexample was found later in $\mathbb{Z}_5 \times \mathbb{Z}_{25}$ [10]; for other counterexamples see e.g., [12] and references there. Apparently periodic decompositions are also quasi-periodic, thus "good" groups are quasi-periodic. A few classes of non-cyclic quasi-periodic Abelian p-groups are given in [12,17]. It is also shown in [12, Theorem 9] that $\mathbb{Z}_{pqrst}$ with p, q, r, s, t being distinct prime numbers is quasi-periodic. To the author's knowledge, the strongest result so far is [19, Theorem 5.13] (which is taken from [18]) for quasi-periodicity of $G \times \mathbb{Z}_q$, where G is a finite Abelian group and q is a prime number that does not divide $|G|$. It is proved there that if (A, B) is a complementing pair in $G \times \mathbb{Z}_q$, then one component is decomposed by the subgroup of order q.

Definition 6. A natural number s is said to be a simple factor of n if $s \mid n$ but $s^2 \nmid n$. A natural number n is called (i) a square-free number if it equals 1 or every prime divisor of it is also a simple factor of it, (ii) a prime power if it has only one prime divisor.

Let p^a be a prime power and n_0 a square-free number, the main purpose of this article is to prove that any complementing pair in a cyclic group of order $p^a n_0$ is quasi-periodic, with one component decomposable by the the subgroup of order p. The proof is by induction and reduction since the presence of the square-free factor n_0 allows us to perform a Tijdeman decomposition. We also give an explicit example to show that $\mathbb{Z}_{72}$ is the smallest cyclic group that fails to have the strong Tijdeman property (which will be introduced in the last part).

2. Preliminaries

Let S be a subset of an additive Abelian group G, denote the difference set of S as the simple set

$$\Delta S = \left\{ s - s' : s, s' \in S,\ s \neq s' \right\},$$

where $s - s'$ is computed within G.

Lemma 1. *Let A, B be subsets of an additive Abelian group G. Then*

$$A + B = A \oplus B \quad \Leftrightarrow \quad \Delta A \cap \Delta B = \emptyset.$$

Proof. Clearly $A + B = A \oplus B$ if and only if the map

$$(a, b) \mapsto a + b, \quad a \in A,\ b \in B,$$

is injective. This map is not injective if and only if there exist distinct $a, a' \in A$ and $b, b' \in B$ with $a + b = a' + b'$, i.e., $a - a' = b' - b$, which means $\Delta A \cap \Delta B \neq \emptyset$. $\square$

Let us write $\gcd(a,b)$ and $\gcd(A)$ for the greatest common divisor of $a,b \in \mathbb{N}$ and elements in $A \subseteq \mathbb{Z}_n$ respectively. If $p \mid n$, then we use the notation $\langle p \rangle$ for the subgroup generated by p (thus of order n/p) in $\mathbb{Z}_n$. Clearly $A \subseteq \langle p \rangle$ if and only if $p \mid \gcd(A)$.

Let A be a subset of an additive Abelian group G. To avoid confusion, we shall only use the notation

$$mA = \big\{ \underbrace{a + \ldots + a}_{m \text{ times}} : a \in A \big\},$$

if the lifting $A \to mA$ is injective. Here the addition is computed within G.

Similarly if G is $\mathbb{Z}$ or $\mathbb{Z}_n$, then

$$A_m = \big\{ a \bmod m : a \in A \big\} \subseteq \mathbb{Z}_m,$$

is used only if the projection $A \to A_m$ is injective.

Proposition 1 [20]. *Let G be $\mathbb{Z}$ or $\mathbb{Z}_n$, and B a finite subset of G. If (A,B) is a complementing pair in G, and $h \in \mathbb{N}$ with $\gcd(h,|B|) = 1$, then (A,hB) is still a complementing pair in G.*

Proof. The case of $G = \mathbb{Z}$ is shown in [20, Theorem 1]. The case of $G = \mathbb{Z}_n$ then follows by considering the tiling $n\mathbb{Z} \oplus A \oplus B$. $\square$

One should also notice the subtle difference concerning divisions in this proposition. If $G = \mathbb{Z}$ and $h \mid \gcd(B)$, then $(A, h^{-1}B)$ is still a complementing pair, but when G is finite, then the same need not hold unless $\gcd(h,n) = 1$, e.g., if $B = \{0,1,2\}$, $B' = 2B = \{0,2,4\}$, then $\mathbb{Z} = 3\mathbb{Z} \oplus B = 3\mathbb{Z} \oplus B'$. However, if $A' = \{0,1\}$, then $\mathbb{Z}_6 = A' \oplus B'$, but (A',B) is not a complementing pair in $\mathbb{Z}_6$.

Lemma 2. *Let (A,B) be a complementing pair in $\mathbb{Z}_n$, and $p \mid n$. If $B \subseteq \langle p \rangle$, then B is tiling in $\langle p \rangle$, and A can be partitioned into p subsets $A^{(0)}, \ldots, A^{(p-1)}$ so that for each $k \in \mathbb{Z}_p$, we have $A^{(k)} = s_k + C^{(k)}$, where we can choose $S = \{s_0, \ldots, s_{p-1}\}$ to be any set that complements $\langle p \rangle$ in $\mathbb{Z}_n$. Each $(C^{(k)}, B)$ is a complementing pair in $\langle p \rangle$ regardless how S is chosen (despite that $C^{(k)}$ changes with s_k). In particular, $A^{(0)}, \ldots, A^{(p-1)}$ do not depend on B.*

Proof. For each $k \in \mathbb{Z}_p$, set $A^{(k)} = A \cap (k + \langle p \rangle)$, and $a_k = |A^{(k)}|$. By this construction $A^{(k)}$ does not depend on B, and clearly $A^{(0)}, \ldots, A^{(p-1)}$ form a partition of A.

Applying the projection $g \mapsto g \bmod p$ $(g \in \mathbb{Z}_n)$ on the decomposition

$$\mathbb{Z}_n = A \oplus B = \big(A^{(0)} \cup \cdots \cup A^{(p-1)}\big) \oplus B,$$

we obtain

$$\mathbb{Z}_p^{\otimes \frac{n}{p}} = \left(\{0\}^{\otimes a_0} \cup \cdots \cup \{p-1\}^{\otimes a_{p-1}}\right) \oplus \{0\}^{\otimes b}.$$

where $b = |B|$. Comparing both equations we see that

$$a_0 = \cdots = a_{p-1} = \frac{n}{bp},$$

and $A^{(k)} \oplus B = k + \langle p \rangle$, thus $(A^{(k)} - k, B)$ is a complementing pair in $\langle p \rangle$.

Now if $(S, \langle p \rangle)$ is a complementing pair in $\mathbb{Z}_n$, then we set $s_k = S \cap (k + \langle p \rangle)$, and $C^{(k)} = A^{(k)} - s_k$. By this construction we have $C^{(k)} \subseteq \langle p \rangle$, and

$$\Delta C^{(k)} = \Delta(A^{(k)} - s_k) = \Delta(A^{(k)} - k),$$

therefore by Lemma 1 we can conclude that $(C^{(k)}, B)$ is also a complementing pair in $\langle p \rangle$. $\square$

3. Main results

3.1. Quasi-periodicity. Since some multi-mappings are involved here let us further agree on the following notation: if $A \subset \mathbb{Z}_n$ with $p \mid \gcd(A)$, then $p^{-1}A$ means the unique set $A' \subseteq \mathbb{Z}_{\lfloor n/p \rfloor}$ ($\lfloor n/p \rfloor$ is the largest integer not exceeding n/p) such that $pA' = A$. For example, if $A = 2B = \{0, 2, 4\} \subset \mathbb{Z}_6$, then B can be any of $\{0,1,2\}, \{0,4,2\}, \{0,1,5\}, \{0,4,5\}$, but by $2^{-1}A$ we mean the set $\{0,1,2\}$.

LEMMA 3. *If $n = qm$, and $B \subseteq \mathbb{Z}_n$, then the lifting $b \mapsto qb \bmod n$ is injective on B if and only if the projection $b \mapsto b \bmod m$ is injective on B, and in such case we have $q^{-1}(qB) = B_m$.*

PROOF. The lifting is not injective if and only there are distinct $b, b' \in B$ such that

$$q(b - b') \equiv 0 \pmod{n},$$

while the projection is not injective if and only there are distinct $b, b' \in B$ such that $(b - b') \equiv 0 \pmod{m}$, and both lead to the same equivalent condition that $\Delta B \cap \langle m \rangle \neq \emptyset$, which shows that their injectivities are equivalent.

Suppose now that they are injective, then writing every element $b \in B$ into the form $b = x_b m + y_b$, with $x_b \in \mathbb{Z}_q$ and $y_b \in \mathbb{Z}_m$ we obtain $q^{-1}(qB) = B_m = \{y_b : b \in B\}$. $\square$

THEOREM 1. *If $n = p^a n_0$ where n_0 is a square-free number, p is a prime number that does not divide n_0 and $a \in \mathbb{N}$, then $\mathbb{Z}_n$ is quasi-periodic.*

Proof. Let H be the order p subgroup in $\mathbb{Z}_n$, and $\tau(n_0)$ the number of prime divisors of n_0. We shall establish the following statement by induction on $\tau(n_0)$:

(Q) Any complementing pair in $\mathbb{Z}_n$ is quasi-periodic, with one component decomposable by H.

If $\tau(n_0) = 0$, then $n_0 = 1$ and (Q) holds trivially since $\mathbb{Z}_{p^a}$ is a "good" group and thus periodic, and clearly the periodic component in any complementing pair is decomposable by H.

Suppose (Q) holds for $\tau(n_0) < t$, and consider now the case of $\tau(n_0) = t$. Let (A, B) be a complementing pair in $\mathbb{Z}_n$, and let q be a prime divisor of n_0. Denote

$$m = \frac{n}{q} = p^a \cdot \frac{n_0}{q}.$$

Since q is a simple factor of n, it can divide precisely one of $|A|$, $|B|$, say $|A|$ without loss of generality. By Proposition 1, (A, qB) is then also a complementing pair in $\mathbb{Z}_n$. Consequently by Lemma 2, A can be partitioned into q subsets $A^{(0)}, \dots, A^{(q-1)}$ where

$$A^{(k)} = k + C^{(k)},$$

for every $k \in \mathbb{Z}_q$, and each $(C^{(k)}, qB)$ is a complementing pair in $\langle q \rangle$. By Lemma 3 each $(q^{-1}C^{(k)}, B_m)$ is also a complementing pair in $\mathbb{Z}_m$. In particular, since $\tau(n_0/q) = t - 1 < t$, the induction assumption applies and (Q) holds in $\mathbb{Z}_m$. Notice also that the order p subgroup in $\mathbb{Z}_m$ is $q^{-1}H$, and let us further write out elements in H as $\{h_0, \dots, h_{p-1}\}$ with

$$h_j = \frac{jn}{p} = jp^{a-1}n_0,$$

for each $j \in \mathbb{Z}_p$.

If B_m is decomposable by $q^{-1}H$, then it can be partitioned into p equal-sized subsets $B_m^{(0)}, \dots, B_m^{(p-1)}$ so that

$$q^{-1}C^{(k)} \oplus B_m^{(j)} = q^{-1}h_j + q^{-1}C^{(k)} \oplus B_m^{(0)},$$

holds for each $j \in \mathbb{Z}_p$. Consequently with $B^{(j)} = qB_m^{(j)}$ we get

$$C^{(k)} \oplus B^{(j)} = h_j + C^{(k)} \oplus B^{(0)},$$

and therefore

$$A \oplus B^{(j)} = \bigcup_{k \in \mathbb{Z}_q} (k + C^{(k)}) \oplus B^{(j)} = h_j + \bigcup_{k \in \mathbb{Z}_q} (k + C^{(k)}) \oplus B^{(0)} = h_j + A \oplus B^{(0)},$$

holds for each $j \in \mathbb{Z}_p$. And clearly $B^{(0)}, \dots, B^{(p-1)}$ is an equal-sized partition of B since

$$|B^{(j)}| = |B_m^{(j)}| = \frac{1}{p}|B_m| = \frac{1}{p}|B|,$$

holds for each $j \in \mathbb{Z}_p$. This shows quasi-periodicity of (A, B) with B decomposable by H.

If B_m is not decomposable by $q^{-1}H$, then by the induction assumption each $q^{-1}C^{(k)}$ must be decomposable by $q^{-1}H$, and thus it can be partitioned into p equal-sized subsets $q^{-1}C^{(k,0)}, \dots, q^{-1}C^{(k,p-1)}$ so that

$$q^{-1}C^{(k,j)} \oplus B_m = q^{-1}h_j + q^{-1}C^{(k,0)} \oplus B_m,$$

holds for each $j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_q$. Consequently with $C^{(k,j)} = q(q^{-1}C^{(k,j)})$ we get

$$C^{(k,j)} \oplus B = h_j + C^{(k,0)} \oplus B,$$

and therefore with

$$A^{(j)} = \bigcup_{k \in \mathbb{Z}_q} k + C^{(k,j)},$$

we obtain similarly an equal-sized partition of A and

$$A^{(j)} \oplus B = \bigcup_{k \in \mathbb{Z}_q} (k + C^{(k,j)}) \oplus B = h_j + \bigcup_{k \in \mathbb{Z}_q} (k + C^{(k,0)}) \oplus B = h_j + A^{(0)} \oplus B,$$

which again shows quasi-periodicity of (A, B) with A decomposable by H. $\square$

Remark. Using the same setting and notations as in the proof of Theorem 1, as $\mathbb{Z}_{p^a n_0} = \mathbb{Z}_m \times ===== mathbbZ_q$, we can see by [19, Theorem 5.13] that A is decomposable by the subgroup of order q, while Theorem 1 says that one of A, B is decomposable by the subgroup of order p.

3.2. An example concerning the strong Tijdeman property.

Definition 7. A subset of $\mathbb{Z}_n$ is said to be normalized if it contains 0. A normalized tiling set A with $\gcd(A) = 1$ is said to have the

- strong Tijdeman property: If all its normalized tiling complements are included in the same proper subgroup (which depends only on A).
- Tijdeman property: If each of its normalized tiling complement is included in some proper subgroup (which may vary for different tiling complements).
- weak Tijdeman property: If it has a normalized tiling complement that is included in some proper subgroup.

$\mathbb{Z}_n$ is said to have the Tijdeman property (resp. strong/weak Tijdeman property) if all such tiling sets A have the the Tijdeman property (resp. strong/weak Tijdeman property).

The normalization condition is vital and shall not be dropped or overlooked.

The Tijdeman property is also called the Rédei property in some literature. Tijdeman conjectured that all finite cyclic groups have the Tijdeman property, while Coven and Meyerowitz noticed in [2] that decompositions constructed in [15] fail to have even the weak Tijdeman property. The smallest n for such counterexamples is $n = 5400$, a decomposition that fails to have the Tijdeman property with $n = 900$ is available in [5]. On the other hand, [6] showed that all cyclic "good" groups possess the strong Tijdeman property, and it is also shown in [11] (alternatively see [2, Lemma 2.4] or [16, Theorem 9.3.1]) that $\mathbb{Z}_{p^a q^b}$ with p, q being distinct prime numbers and $a, b \in \mathbb{N}$ has the Tijdeman property.

It was mentioned in both [20] and [6] without details that the Tijdeman property would imply quasi-periodicity. We are not able to locate a source for a proof (except a partial one in [12, Theorem 8] for square-free n), neither can we produce one unfortunately.

We conclude this article by providing an explicit counterexample to the strong Tijdeman property in $\mathbb{Z}_{72}$, so that besides those counterexamples to the Tijdeman property and the weak Tijdeman property in [5] and [15] respectively, we now have a full collection of counterexamples to each of the properties in Definition 7. It is easy to see from the list of "good" groups that $\mathbb{Z}_{72}$ is also the smallest cyclic group that fails to have the strong Tijdeman property.

If $A \subset \mathbb{Z}_n$, then define

$$F_A(z) = \sum_{a \in A} z^a.$$

Denote Φ_m as the m-th cyclotomic polynomial, if (A, B) is a complementing pair in $\mathbb{Z}_n$, then

(1)
$$F_A(z) \cdot F_B(z) \equiv F_{\mathbb{Z}_n}(z) \equiv 1 + z + \cdots + z^{n-1} \equiv \prod_{\substack{m \mid n \\ m \neq 1}} \Phi_m(z) \pmod{z^n - 1}.$$

If (A, B) is a complementing pair in $\mathbb{Z}_{p^3 q^2}$ (p, q are distinct prime numbers) with $0 \in A \cap B$ and $\gcd(A) = 1$, then a simple but tedious (thus omitted here) combinatorial analysis indicates that to show the strong Tijdeman property fails, one should look for B with $|B| = pq$ and $\Phi_{p^2}, \Phi_{q^2} \mid F_B$. By a computer aided verification we obtain the following:

Example 2. If

$$A = \{0, 1, 4, 5, 8, 9, 36, 37, 40, 41, 44, 45\},$$
$$A' = \{0, 1, 2, 9, 10, 11, 36, 37, 38, 45, 46, 47\},$$

so that $0 \in A, A'$, $\gcd(A) = \gcd(A') = 1$ with

$$F_A(z) = \Phi_2(z)\Phi_3(z)\Phi_6(z)\Phi_8(z)\Phi_{12}(z)\Phi_{24}(z)\Phi_{72}(z),$$
$$F_{A'}(z) = \Phi_2(z)\Phi_3(z)\Phi_6(z)\Phi_8(z)\Phi_{18}(z)\Phi_{24}(z)\Phi_{72}(z),$$

and

$$B = \{0, 3, 6, 18, 21, 24\}, \quad B' = \{0, 2, 12, 14, 24, 26\},$$

so that $0 \in B, B'$ with

$$F_B(z) = \Phi_4(z)\Phi_9(z)\Phi_{12}(z)\Phi_{36}(z), \quad F_{B'}(z) = \Phi_4(z)\Phi_9(z)\Phi_{18}(z)\Phi_{36}(z),$$

then

$$\mathbb{Z}_{72} = A \oplus B = A \oplus B' = A' \oplus B = A' \oplus B',$$

but clearly $B \subset \langle 3 \rangle$ while $B' \subset \langle 2 \rangle$.

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