Irregular Orthonormal Gabor Basis in Finite Dimensions and Applications

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***Abstract*—Gabor analysis is playing an important role in time-frequency analysis for the last 60 years. The fundamental concept of Gabor frames has become a focus also in the finite dimensional setting. Here, we study which sets of discrete time frequency shifts admit a proper choice of window vector, so that they generate an orthonormal Gabor basis in the underlying finite dimensional vector space. We fully characterize such time-frequency shift sets in the case that the vector space dimension is a prime number.**

Keywords: Unitary Gabor Matrices; Discrete Gabor Analysis; Bi-unimodular Sequences; CAZAC Sequences; Sampling of Operators

2010 MSC: 42A16, 42A99

I. Introduction

Since its debut in a 1946 seminal paper ([5]), Gabor analysis has evolved quickly and has become an indispensable tool in time-frequency analysis (for example, see [4, 6, 8]). The fundamental idea behind it, i.e., to deompose a communications channel into small subchannels supported in rectangular boxes on the time-frequency plane, can also be considered for finite dimensional vector spaces (see [11]). Finite dimensional Gabor frames feature many nice properties and find their application in various areas, see [10, 1, 3, 12, 14, 15, 16].

Nevertheless, many numerical aspects of finite dimensional Gabor frames are barely studied yet. This article aims to explore this fairly blank area. Here we focus on constructing orthonormal Gabor basis on irregular subsets from the full lattice of time frequency shifts, i.e., we aim to answer the following question: Which subsets of the discrete time frequency plane can actually admit a proper choice of window vector, so that together they form a orthonormal basis (i.e., orthonormal Gabor basis) for the underlying finite dimensional vector spaces.

We will give a full characterization of the scenario when the dimension is a prime number. This has a direct application to the operator identification problem (see [13]) which the authors are particularly interested in, and helps to understand the structure of tight subframes from the full discrete Gabor frame.

To better describe our setting, as well as to facilitate the subsequent content, we first introduce our notation and definitions.

Let N be a prime number, and let $\mathbb{C}^N$ be the space of N dimensional complex vectors, equipped with the Euclidean norm $\|\cdot\|$. Let $\mathbb{C}^{N\times N}$ be the space of $N\times N$ matrices equipped with the Hilbert-Schmidt norm $\|\cdot\|_{HS}$, i.e., the ℓ^2 norm of its entries. The inner products on both spaces are denoted as $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle x,y\rangle = y^* x, \quad x,y\in\mathbb{C}^N,$$
$$\langle A,B\rangle = \mathrm{tr}(AB^*), \quad A,B\in\mathbb{C}^{N\times N},$$

where A^* (resp. x^*) denotes the adjoint of A (resp. x), and $\mathrm{tr}(A)$ is its trace.

Let

$$w = e^{\frac{2\pi i}{N}},$$

be the first primitive root of unity, and let

$$W = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w^1 & \dots & (w^{N-1})^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (w)^{N-2} & \dots & (w^{N-1})^{N-2} \\ 1 & (w)^{N-1} & \dots & (w^{N-1})^{N-1} \end{pmatrix},$$

be the unitary inverse discrete Fourier matrix, i.e.,

$$W_{mn} = \frac{1}{\sqrt{N}} w^{(m-1)(n-1)}. \tag{1}$$

For $j,k\in\mathbb{Z}$, the matrices

$$M^j = \begin{pmatrix} 1 & & & & \\ & w & & & \\ & & \ddots & & \\ & & & w^{N-2} & \\ & & & & w^{N-1} \end{pmatrix}^j,$$

$$T^k = \begin{pmatrix} 0 & & & & 1 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}^k,$$

act as frequency shifts (modulation), and time shifts (translation) on $\mathbb{C}^N$. Similar to the continuous case, these discrete time frequency shifts are connected via the discrete Fourier transform as

$$T^k = WM^{-k}W^*.$$

With this setting, one can define the (full) discrete Gabor system $\{M^jT^kd\}_{j,k=0,1,2,\dots N-1}$ with respect to any window vector $d \in \mathbb{C}^N$. Regardless of the choice of d, the set $\{M^jT^kd\}_{j,k=0,1,2,\dots N-1}$ is always a tight frame [11].

Consider the group $\mathbb{Z}_N \times \mathbb{Z}_N$. If Γ is a subset of $\mathbb{Z}_N \times \mathbb{Z}_N$, then we use the notation (d,Γ) to denote the set of vectors consists of discrete time frequency shifts supported on Γ, and applied to d, i.e.,

$$(d,\Gamma) = \{M^{m_1}T^{n_1}d, M^{m_2}T^{n_2}d, \dots\}, \tag{2}$$

where $(m_1,n_1),\ (m_2,n_2),\ \dots \in \Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$.

Our main problem now posed as follows. Given

$$\Gamma = \{(j_1,k_1),(j_2,k_2),\dots,(j_N,k_N)\} \subset \mathbb{Z}_N \times \mathbb{Z}_N,$$

a proper subset of $\mathbb{Z}_N \times \mathbb{Z}_N$ with cardinality N. What conditions does Γ need to satisfy so that one can find a $d \in \mathbb{C}^N$ for (d,Γ) to be an orthonormal basis for $\mathbb{C}^N$?

The motivating application is, as described in [13], the identification problem for underspread operators that can be reconstructed by means of so-called operator sampling. The design of an identifier mandates the construction of a vector d in C^N so that (d,Γ) is a Riesz basis. The set Γ is given and decodes the bandlimitation (spreading support) of the unknown operator. Stability of the operator identification procedure profits from (d,Γ) being close to an orthonormal system.

The remaining part of this article is organized as follows. In the next section we establish properties of discrete time frequency shifts and discrete Gabor frames that are necessary for our proof, in Section III we prove that the answer to the above question is the following.

Theorem 1. *There exists $d \in \mathbb{C}^N$ for (d,Γ) to be an orthonormal basis of $\mathbb{C}^N$ if and only if there is a proper (and non-trivial) subgroup V of $\mathbb{Z}_N \times \mathbb{Z}_N$ such that*

$$\mathbb{Z}_N \times \mathbb{Z}_N = \Gamma + V = \{(j+j', k+k') : (j,k) \in \Gamma, (j',k') \in V\}.$$

In other words, one can find $d \in \mathbb{C}^N$ for (d,Γ) to be an orthonormal basis of $\mathbb{C}^N$ if and only if Γ contains precisely one element from each coset of a subgroup V.

II. Properties of Discrete Time Frequency Shifts and Discrete Gabor Frames

Here, we list of few properties needed to prove Theorem 1. We omit proof details since these properties can be verified directly.

Lemma II.1 (Frame Properties) *The following holds:*

1) *$\{\frac{1}{\sqrt{N}}M^jT^k\}_{j,k=0,1,2,\dots,N-1}$ is an orthonormal basis for the matrix space $\mathbb{C}^{N\times N}$.*
2) *$\{M^jT^kd\}_{j,k=0,1,2,\dots,N-1}$ is a tight frame on $\mathbb{C}^N$ with frame constant $N\|d\|^2$ for any $d \in \mathbb{C}^N$.*

Lemma II.2 (Commutativity) *M^j and T^k commute up to a phase factor, i.e.,*

$$M^jT^k = w^{jk}T^kM^j.$$

Consequently, if

$$kj' \equiv jk' \pmod N, \tag{3}$$

then M^jT^k and $M^{j'}T^{k'}$ commute, and can be simultaneously diagonalized.

The finite Heisenberg group of order N^3 consists of tuples (h,j,k) with $h,j,k \in \mathbb{Z}_N$ and the group law

$$(h,j,k) \odot (h',j',k') \mapsto (h+h'-kj', j+j', k+k'),$$

hence it can be viewed as the semiproduct

$$(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes_\phi \mathbb{Z}_N,$$

where

$$\phi : \mathbb{Z}_N \mapsto \mathrm{Aut}(\mathbb{Z}_N \times \mathbb{Z}_N),$$

is given as

$$(\phi(k))((h',j')) = (h'-kj', j').$$

One may verify that since N is prime, each $\phi(k)$ is indeed a group automorphism of $\mathbb{Z}_N \times \mathbb{Z}_N$, and thus the semiproduct is well defined. A concise list of properties of the finite Heisenberg group can be found in [9].

It is easy to see using Lemma II.2 that

$$\rho : (h,j,k) \mapsto w^hM^jT^k,$$

is a representation of this finite group. Denote $\mathbb{H}$ as the above group $\{w^hM^jT^k\}_{h,j,k=0,1,2,\dots,N-1}$. Then by Lemma II.2, the center of $\mathbb{H}$ is

$$Z(\mathbb{H}) = \{w^hI\}_{h=0,1,2,\dots,N-1} = \{\rho(h,0,0)\}_{h=0,1,2,\dots,N-1}.$$

Let us denote $\tilde{F}$ as the quotient group

$$\tilde{F} = \quad \mathbb{H}/Z(\mathbb{H}) \quad \cong \quad \mathbb{Z}_N \times \mathbb{Z}_N.$$

and use $\odot$ to denote the group operation on $\tilde{F}$.

Let

$$\tilde{M}^j\tilde{T}^k = \{w^hM^jT^k\}_{h=0,1,2,\dots,N-1},$$

then we have

$$\tilde{F} = \{\tilde{M}^j\tilde{T}^k\}_{j,k=0,1,2\dots N-1},$$

$$\tilde{M}^j\tilde{T}^k \odot \tilde{M}^{j'}\tilde{T}^{k'} = \tilde{M}^{j+j'}\tilde{T}^{k+k'}.$$

Lemma II.3 (Subgroup Structure) *If N is a prime number, then the only subgroups in $\mathbb{Z}_N \times \mathbb{Z}_N$ are*

$$V_s = \begin{cases} \{(ks, k)\}_{k=0,1,2,\dots N-1}, & s = 0, 1, 2, \dots N-1 \\ \{(j, 0)\}_{j=0,1,2,\dots N-1}, & s = \infty \end{cases}$$

and they pairwise intersect trivially.

If Γ is a subset of $\mathbb{Z}_N \times \mathbb{Z}_N$, then we use the notation $\tilde{\Gamma}$ to denote the corresponding subset in $\tilde{F}$, i.e.,

$$\tilde{\Gamma} = \{\tilde{M}^j \tilde{T}^k : (j, k) \in \Gamma\},$$

and vice versa. In particular, V_s would denote the following subsets

$$\tilde{V}_s = \begin{cases} \{\tilde{M}^{ks} \tilde{T}^k\}_{k=0,1,2,\dots N-1}, & s = 0, 1, 2, \dots N-1 \\ \{\tilde{M}^j\}_{j=0,1,2,\dots N-1}, & s = \infty \end{cases},$$

and one may also verify using Lemma II.2 that for fixed s, members in $\{M^{ks} T^k\}_{k=0,1,2,\dots,N-1}$ commute.

Lemma II.4 (Eigenstructure) *Let D be the diagonal matrix in which the ℓ-th entry on the main diagonal is*

$$D_{\ell\ell} = w^{0+1+2+\dots+(\ell-1)},$$

then for N odd, we have:

1) *The eigenvectors of $M^1, M^2, \dots, M^{N-1}$ are precisely the Euclidean column basis.*
2) *For $s = 0, 1, 2, \dots, N-1$ and $k = 1, 2, \dots, N-1$, each $M^{ks} T^k \in V_s$ can be diagonalized as*

$$M^{ks} T^k = w^{-\frac{k(k-1)s}{2}} D^s W M^{-k} W^* (D^s)^*.$$

Let x be a vector of unit length. We introduce the notation

$$P_x = xx^*$$

to denote the orthogonal projection onto the span of the vector $x \in \mathbb{C}^N$.

Lemma II.5 *The following holds:*

1)
$$\langle M^j T^k, P_x \rangle = \langle M^j T^k x, x \rangle.$$

2) *For any $A \in \mathbb{C}^{N \times N}$, one has*

$$M^j A (M^j)^* = N A \circ P_{u_j},$$

where $\circ$ denotes the Hadamard product.

Lemma II.6 *If $R \in \mathbb{N}$ and $R < N$, then for*

$$\{j_1, j_2, \dots, j_R\} \subset \{1, 2, \dots, N\},$$

and $a_1, a_2, \dots, a_R \in \mathbb{N}$, we have

1)
$$a_1 w^{j_1} + a_2 w^{j_2} + \dots + a_R w^{j_R} \neq 0.$$

2) *Each entry on the main diagonal of $a_1 P_{u_{j_1}} + a_2 P_{u_{j_2}} + \dots + a_R P_{u_{j_R}}$ is $(a_1 + a_2 + \dots + a_R)/N$.*
3) *Each off diagonal entry of $a_1 P_{u_{j_1}} + a_2 P_{u_{j_2}} + \dots + a_R P_{u_{j_R}}$ is non zero.*

Denote the columns in W (as defined in (1)) as $u_0, u_1, \dots, u_{N-1}$.

Lemma II.7 *Assume N is odd and D as given in Lemma II.4, then for any $s, \ell = 0, 1, 2, \dots, N-1$, if $x = D^s u_\ell$ is a column of $D^s W$, then $M^j x$ and $T^k x$ are still columns (up to a phase difference) in $D^s W$ with explicit formula:*

$$M^j D^s u_\ell = D^s u_{\ell+j},$$

$$T^k D^s u_\ell = w^{-\frac{k(k-1)s+2k\ell}{2}} D^s u_{\ell-ks},$$

where (throughout this paper) addition and multiplication of indices are understood as operations in the finite field $\mathbb{Z}_N$.

Lemma II.8 *If x is a unit eigenvector of any element in V_s for some fixed $s \in \{0, 1, 2, \dots, N-1, \infty\}$, then*

$$P_x \perp (V_{s'} \setminus \{I\}) \quad \textit{for any } s' \neq s.$$

III. Sketch of Proof for Theorem 1

Using the notations and concepts from the last section, we can restate our result alternatively as follows,

Theorem 1. *There exists $d \in \mathbb{C}^N$ for (d, Γ) to be an orthonormal basis of $\mathbb{C}^N$ if and only if there is a proper (and non-trivial) subgroup V_s of $\mathbb{Z}_N \times \mathbb{Z}_N$ such that*

$$\mathbb{Z}_N \times \mathbb{Z}_N = \Gamma + V_s = \{(j+j', k+k') : (j,k) \in \Gamma, (j', k') \in V_s\}. \tag{4}$$

And in such cases, it suffices to choose d to be any shared unit eigenvector of the members in V_s.

Proof. We give a sketch of proof by listing the main steps below.

Lemma II.3 listed all subgroups to be considered.

For convenience, denote the elements in Γ as

$$\Gamma = \{(j_1, k_1), (j_2, k_2), \dots, (j_N, k_N)\},$$

and set

$$G = (M^{j_1} T^{k_1} d, \quad M^{j_2} T^{k_2} d, \quad \dots, \quad M^{j_N} T^{k_N} d),$$

$$G_s = \begin{pmatrix} d & M^s T^1 d, & M^{2s} T^2 d, & \dots, & M^{(N-1)s} T^{N-1} d \end{pmatrix}$$

for $s \in \{0, 1, 2, \dots, N-1\}$ and

$$G_\infty = (d \quad M^1 d, \quad M^2 d, \quad \dots, \quad M^{N-1} d).$$

In other words, G is a matrix form of (d, Γ) and G_s is a matrix form of (d, V_s).

Sufficiency: Suppose that for given Γ, there exists some V_s that satisfies (4).

Choose d to be a unit eigenvector of members in V_s, then clearly all columns in G are of unit norm, thus it suffice to show that they are pairwise orthogonal.

Define the first order difference set $\Delta\tilde{\Gamma}$ of $\tilde{\Gamma}$ to be

$$\Delta\tilde{\Gamma} = \{\tilde{M}^{j-j'}\tilde{T}^{k-k'} : \tilde{M}^{j}\tilde{T}^{k}, \tilde{M}^{j'}\tilde{T}^{k'} \in \Gamma\}.$$

It is obvious from (4) that

$$I \notin \Delta\tilde{\Gamma}.$$

By Lemma II.5 (1), to show columns in G are pairwise orthogonal, it suffice to establish

$$P_d \perp \Delta\tilde{\Gamma}.$$

To prove this, we observe that (4) necessarily implies

$$\Delta\tilde{\Gamma} \cap \tilde{V}_s = \emptyset. \tag{5}$$

Indeed, by a counting argument we must have that if $\tilde{M}^m\tilde{T}^n$ and $\tilde{M}^{m'}\tilde{T}^{n'}$ are distinct elements in $\tilde{V}_s$, then

$$(\tilde{M}^m\tilde{T}^n \odot \tilde{\Gamma}) \cap (\tilde{M}^{m'}\tilde{T}^{n'} \odot \tilde{\Gamma}) = \emptyset, \tag{6}$$

while if (5) does not hold and

$$(\Delta\tilde{\Gamma} \cap \tilde{V}_s) \ni \tilde{M}^a\tilde{T}^b,$$

then since $I \notin \Delta\tilde{\Gamma}$, we have

$$\tilde{M}^a\tilde{T}^b \neq \tilde{M}^0\tilde{T}^0,$$

while by definition of $\Delta\tilde{\Gamma}$ we would have

$$(M^0\tilde{T}^0 \odot \tilde{\Gamma}) \cap (\tilde{M}^a\tilde{T}^b \odot \tilde{\Gamma}) \neq \emptyset,$$

which contradicts (6).

Now that we have established (5), it follows that

$$\Delta\tilde{\Gamma} \subseteq (\tilde{F} \setminus \tilde{V}_s) = \bigcup_{\substack{s' \neq s \\ s' \in \{0,1,2,\ldots,N-1,\infty\}}} \tilde{V}_{s'} \setminus \{I\},$$

hence by Lemma II.8 we can conclude that

$$P_d \perp \Delta\tilde{\Gamma}.$$

To prove necessity, we first assume the contrary, i.e., suppose there exists d such that (d, Γ) is an orthonormal basis of $\mathbb{C}^N$ and (4) fails for all $s \in \{0, 1, 2, \ldots, N-1, \infty\}$. Then we show that (d, Γ) being an orthonormal basis of $\mathbb{C}^N$ while (4) fails implies that (d, V_s) (and consequently G_s) is an orthonormal basis of $\mathbb{C}^N$ for all $s \in \{0, 1, 2, \ldots, N-1, \infty\}$, but then Lemma II.5 would imply that

$$P_d \perp (F \setminus \{I\}).$$

By Lemma II.1, this would imply that P_d is a constant multiple of the identity, which is impossible since P_d only has rank 1. This way we derive a contradiction.

Let us first consider a fixed $s \in \{0, 1, 2, \ldots, N-1\}$.

It is easy to verify that

$$\tilde{F} = \tilde{V}_\infty \odot \tilde{V}_s.$$

Now each $\tilde{M}^j\tilde{T}^k \odot \tilde{V}_s$ is a coset of $\tilde{V}_s$, therefore both $\tilde{V}_\infty \odot \tilde{V}_s$ and $\tilde{\Gamma} \odot \tilde{V}_s$ are unions of cosets of $\tilde{V}_s$. Recall that by Lemma II.3, $\tilde{V}_s$ is a subgroup, thus there are only N distinct cosets. Hence we can find a map f from $\tilde{\Gamma}$ into $\tilde{V}_\infty$ such that

$$\tilde{M}^{j_\ell}\tilde{T}^{k_\ell} \odot \tilde{V}_s = f(\tilde{M}^{j_\ell}\tilde{T}^{k_\ell}) \odot \tilde{V}_s,$$

for all $\ell \in \{1, 2, \ldots, N\}$.

Denote

$$J = \{f(\tilde{M}^{j_\ell}\tilde{T}^{k_\ell}) : \tilde{M}^{j_\ell}\tilde{T}^{k_\ell} \in \tilde{\Gamma}\},$$

i.e., the range of f, and set

$$R = |J|.$$

Since (4) is not satisfied, we have

$$R < N.$$

Let us write the elements in J as $\tilde{M}^{\ell_1}, \tilde{M}^{\ell_2}, \ldots, \tilde{M}^{\ell_R}$, and define

$$a_m = |f^{-1}(\tilde{M}^{\ell_m})|,$$

for all $m = 1, 2, \ldots, R$.. i.e., a_m counts the number of elements in the pre-image of $\tilde{M}^{\ell_m}$. In particular, it holds that

$$a_1 + a_2 + \ldots + a_R = |\tilde{\Gamma}| = N.$$

By our assumption G is unitary, thus columns in

$$B_s = \left(\begin{array}{c|c|c|c} G & M^sT^1G & \ldots & M^{(N-1)s}T^{N-1}G \end{array}\right),$$

form a tight frame with frame bound N since it contains N unitary matrices collected together.

On the other hand, our previous derivation shows that after permuting the columns and scaling them by a phase factor, B_s can be rearranged into

$$B_s' = \left(\begin{array}{c|c|c} f(M^{j_1}T^{k_1})G_s & \ldots & f(M^{j_N}T^{k_N})G_s \end{array}\right),$$

which is also a tight frame with frame bound N as B_s.

Consequently we get

$$\begin{aligned} NI &= B_s'(B_s')^* \\ &= NG_sG_s^* \circ (a_1P_{u_{\ell_1}} + a_2P_{u_{\ell_2}} + \ldots + a_RP_{u_{\ell_R}}), \end{aligned}$$

where the second equality follows from the definition of $a_1, a_2, \ldots, a_R$ and Lemma II.5.

Now since $R < N$, we use Lemma II.6 to conclude that each off diagonal entry of $a_1P_{u_{j_1}} + a_2P_{u_{j_2}} + \ldots + a_RP_{u_{j_R}}$ is non zero, and each main diagonal entry of it is

$$\frac{1}{N}(a_1 + a_2 + \ldots + a_R) = 1.$$

Thus comparing both sides of the above equation, we obtain that

$$I = G_sG_s^*,$$

which shows that G_s is unitary, and thus (d, V_s) is unitary.

The case $s = \infty$ can be derived with similar computations, and together these complete the proof. $\square$

Remark 1. Here we give a geometric interpretation of the condition in (4):

Consider a grid that consists of $N \times N$ boxes, each box is mapped to an element in $\tilde{F}$ as follows: counting from left to right, bottom to top, (j, k) maps to the box at the intersection of the $j+1$-st column and the $k+1$-st row. Thus each Γ uniquely maps to a domain consists of N boxes, and shifting Γ by the elements in V_s can be interpreted as translating the domain by a certain number of boxes along a line of slope s.

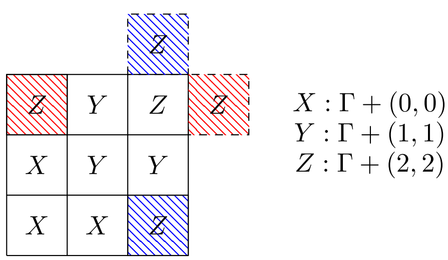
Under this construction, (4) is equivalent to saying that there exists some s, such that shifting the domain along the line of slope s by $0, 1, 2, \ldots, N-1$ boxes and periodizing by the grid leads a tiling of the grid. Below is an example for $N = 3$ with

$$\Gamma = \{(0,0), (0,1), (1,0)\},$$

Each box of the corresponding domain has been marked by X, and the image of each shift of it along the diagonal line ($s = 1$) has been marked Y and Z respectively. The coloring demonstrates the periodization. We see that in this case, these shifts tiles up the original grid, i.e.,

$$\mathbb{Z}_N \times \mathbb{Z}_N = \Gamma + V_1,$$

and choosing d to be a shared unit eigenvector of members in $\tilde{V}_1$ makes (d, Γ) an orthonormal basis of $\mathbb{C}^N$.



Remark 2. A **bi-unimodular** sequence (or a **CAZAC** sequences, CAZAC stands for Constant Amplitude Zero Auto Correlation Sequences) is a sequence that is unimodular before and after applying the discrete Fourier transform. Such sequences are of special interests in engineering and are also connected to the so called cyclic N-roots. See, for example, [7] and [2] for related concepts.

By Lemma II.5 (1), we can see that x is bi-unimodular if and only if P_x is orthogonal to all the pure time shifts $T^1, T^2, \ldots, T^{N-1}$ and pure frequency shifts $M^1, M^2, \ldots, M^{N-1}$. Thus Lemma II.4 and II.8 shows that eigenvectors of $\tilde{V}_s$ (namely columns in $\sqrt{N} D^s W$) for $s = 1, 2, \ldots, N-1$ are biunimodular sequences, and the quotients of their adjacent entries form the classical solutions for unimodular cyclic N-roots (But as is known, these are not the only biunimodular sequences, see examples in [2]).

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